Problem 1. Suppose that a cubic (3-regular) graph $G$ has exactly one edge-colouring with $\chi'(G)$ colours, up to a permutation of the colours. Show that $\chi'(G) = 3$ and that $G$ has exactly three Hamilton cycles.

Problem 2. Show that every red/blue-colouring of the edges of $K_{6n}$ contains $n$ vertex-disjoint triangles with all $3n$ edges of the same colour.

Problem 3. Given two graphs $G$ and $H$, let $R(G, H)$ denote the smallest integer $n$ such that every red/blue-colouring of the edges of $K_n$ contains either a blue copy of $G$ or a red copy of $H$. Determine $R(K_s, P_t)$ for all positive $s$ and $t$, where $P_t$ is the path with $t$ vertices.

Problem 4. Prove that for every tree $T$ and integers $k \geq 2$ and $g \geq 3$, there exists a graph $G$ without cycles of length up to $g$ and such that every $k$-colouring of the edges of $G$ contains a monochromatic copy of $T$.

Problem 5. Let $H$ be the graph with four vertices and five edges. Prove that $\text{ex}(n, H) = \text{ex}(n, K_3)$ for every $n \geq 4$.

Problem 6. Suppose that $v_1, \ldots, v_n$ are distinct unit (that is, of length 1) vectors in $\mathbb{R}^3$. Prove that there are at most $4n^{5/3}$ pairs $\{i, j\} \subseteq \{1, \ldots, n\}$ such that $v_i$ and $v_j$ are orthogonal.

Please do NOT submit written solutions to the following exercises:

Exercise 1. Prove that $R(n \cdot K_2, n \cdot K_2) = 3n - 1$ for every $n$, where $n \cdot K_2$ denotes $n$ independent edges.

Exercise 2. Prove that if $n \geq 5$, then every graph of order $n$ with $\lceil n^2/4 \rceil + 2$ edges contains two triangles with exactly one common vertex.

Exercise 3. Let $G$ be an $n$-vertex graph with $\lceil n^2/4 \rceil - t$ edges and no triangle. Show that one can make $G$ bipartite by deleting from it at most $t$ edges.

Exercise 4. Suppose that $n$, $m$, and $t$ are the numbers of vertices, edges, and triangles (respectively) of a graph. Show that $3t \geq 4m^2/n - mn$. Note that this implies that $\text{ex}(n, K_3) \leq n^2/4$. 