Graph Theory

Homework assignment #4

Due date: Sunday, January 13, 2019

Problem 1. Suppose that a cubic (3-regular) graph G has exactly one edge-colouring with $\chi'(G)$ colours, up to a permutation of the colours. Show that $\chi'(G) = 3$ and that G has exactly three Hamilton cycles.

Problem 2. Show that every red/blue-colouring of the edges of K_{6n} contains n vertex-disjoint triangles with all $3n$ edges of the same colour.

Problem 3. Given two graphs G and H, let $R(G, H)$ denote the smallest integer n such that every red/blue-colouring of the edges of K_n contains either a blue copy of G or a red copy of H. Determine $R(K_s, P_t)$ for all positive s and t, where P_t is the path with t vertices.

Problem 4. Prove that for every tree T and integers $k \geqslant 2$ and $g \geqslant 3$, there exists a graph G without cycles of length up to g and such that every k-colouring of the edges of G contains a monochromatic copy of T.

Problem 5. Let H be the graph with four vertices and five edges. Prove that $\mathrm{ex}(n, H) = \mathrm{ex}(n, K_3)$ for every $n \geqslant 4$.

Problem 6. Suppose that $v_1, \ldots, v_n$ are distinct unit (that is, of length 1) vectors in $\mathbb{R}^3$. Prove that there are at most $4n^{5/3}$ pairs $\{i, j\} \subseteq \{1, \ldots, n\}$ such that v_i and v_j are orthogonal.

Please do NOT submit written solutions to the following exercises:

Exercise 1. Prove that $R(n\cdot K_2, n\cdot K_2) = 3n-1$ for every n, where $n\cdot K_2$ denotes n independent edges.

Exercise 2. Prove that if $n \geqslant 5$, then every graph of order n with $\lfloor n^2/4 \rfloor + 2$ edges contains two triangles with exactly one common vertex.

Exercise 3. Let G be an n-vertex graph with $\lfloor n^2/4 \rfloor - t$ edges and no triangle. Show that one can make G bipartite by deleting from it at most t edges.

Exercise 4. Suppose that n, m, and t are the numbers of vertices, edges, and triangles (respectively) of a graph. Show that $3t \geqslant 4m^2/n - mn$. Note that this implies that $\mathrm{ex}(n, K_3) \leqslant n^2/4$.