Graph Theory

Homework assignment #2

Due date: Sunday, December 6, 2020

Problem 1. Prove that every two paths of maximum length in a connected graph must have a vertex in common.

Problem 2. Prove that a graph is 2-connected if and only if for any three vertices x, y, and z, there is a path from x to z that passes through y.

Problem 3. Let G be a 3-regular graph. Prove that $\kappa(G) = \kappa'(G)$.

Problem 4. Show that every k-connected graph with at least $2k$ vertices contains a cycle of length at least $2k$.

Problem 5. Suppose that a graph G contains two edge-disjoint spanning trees. Show that G contains a spanning Eulerian subgraph, that is, a spanning subgraph that has an Eulerian tour.

Problem 6. Let G be a connected graph with n vertices. Prove that G contains a path of length $\min\{2\delta(G), n-1\}$.

Problem 7. Let G be a non-bipartite graph with n vertices. Show that G has an odd cycle of length at most $\max\big\{3, 2n/\delta(G)\big\}$.

Problem 8. A *tournament* is a complete graph in which each edge uv is given a direction, either from u to v or from v to u. Show that a tournament must contain a Hamiltonian path, that is, a directed path through all the vertices. Must it contain a Hamiltonian cycle?

Please do NOT submit written solutions to the following exercises:

Exercise 1. Let G be a graph and let $A \subseteq V(G)$. Let H be the graph obtained from G by adding to it a new vertex v with $N_H(v) = A$. Show that $\kappa(H) \geqslant \min\{|A|, \kappa(G)\}$.

Exercise 2. Prove that G contains the path of length two as an induced subgraph if and only if G is not a union of vertex-disjoint complete graphs.

Exercise 3. Prove that every connected graph has a vertex that is not a cutvertex.