Graph Theory
Homework assignment #2
Due date: Sunday, December 6, 2020

Problem 1. Prove that every two paths of maximum length in a connected graph must have a vertex in common.

Problem 2. Prove that a graph is 2-connected if and only if for any three vertices $x, y, z$, there is a path from $x$ to $z$ that passes through $y$.

Problem 3. Let $G$ be a 3-regular graph. Prove that $\kappa(G) = \kappa'(G)$.

Problem 4. Show that every $k$-connected graph with at least $2k$ vertices contains a cycle of length at least $2k$.

Problem 5. Suppose that a graph $G$ contains two edge-disjoint spanning trees. Show that $G$ contains a spanning Eulerian subgraph, that is, a spanning subgraph that has an Eulerian tour.

Problem 6. Let $G$ be a connected graph with $n$ vertices. Prove that $G$ contains a path of length $\min\{2\delta(G), n - 1\}$.

Problem 7. Let $G$ be a non-bipartite graph with $n$ vertices. Show that $G$ has an odd cycle of length at most $\max\{3, 2n/\delta(G)\}$.

Problem 8. A tournament is a complete graph in which each edge $uv$ is given a direction, either from $u$ to $v$ or from $v$ to $u$. Show that a tournament must contain a Hamiltonian path, that is, a directed path through all the vertices. Must it contain a Hamiltonian cycle?

Please do NOT submit written solutions to the following exercises:

Exercise 1. Let $G$ be a graph and let $A \subseteq V(G)$. Let $H$ be the graph obtained from $G$ by adding to it a new vertex $v$ with $N_H(v) = A$. Show that $\kappa(H) \geq \min\{|A|, \kappa(G)\}$.

Exercise 2. Prove that $G$ contains the path of length two as an induced subgraph if and only if $G$ is not a union of vertex-disjoint complete graphs.

Exercise 3. Prove that every connected graph has a vertex that is not a cutvertex.