Probabilistic methods in combinatorics

Homework assignment #1

Due date: Monday, April 8, 2019

Problem 1. Let $F$ be a finite collection of binary strings of finite lengths such that no member of $F$ is a prefix of another. Let $N_i$ denote the number of strings of length $i$ in $F$. Prove that

$$\sum_{i=1}^{\infty} \frac{N_i}{2^i} \leq 1.$$ 

Problem 2. Let $k \geq 1$ and suppose that $A$ is a collection of subsets of $[n]$ that does not contain a chain of $k+1$ sets. That is, there are no $A_1, \ldots, A_{k+1} \in A$ such that $A_1 \subseteq \cdots \subseteq A_{k+1}$. Show that $A$ can have at most as many elements as the largest union of $k$ levels of the Boolean lattice, that is,

$$|A| \leq \max \left\{ \sum_{i \in I} \binom{n}{i} : I \subseteq [n] \text{ and } |I| = k \right\}.$$ 

Problem 3. Prove that an arbitrary set of ten points in the plane can be covered by a family of pairwise disjoint unit disks.

Problem 4. Prove that every set of $n$ nonzero integers contains two disjoint sum-free subsets $B_1$ and $B_2$ such that $|B_1| + |B_2| > 2n/3$.

Problem 5. Suppose that $G$ is a graph with no isolated vertices. Show that there exist disjoint independent sets $I$ and $J$ such that

$$|I| + |J| \geq \sum_{v \in V(G)} \frac{2}{\deg_G v + 1}.$$

Problem 6. Show that there is a positive constant $\delta$ such that the following holds. Suppose that $a_1, \ldots, a_n \in \mathbb{R}$ satisfy $a_1^2 + \cdots + a_n^2 = 1$ and that $\varepsilon_1, \ldots, \varepsilon_n$ are independent random variables with $\Pr(\varepsilon_i = 1) = \Pr(\varepsilon_i = -1) = 1/2$ for every $i$. Then

$$\Pr \left( |\varepsilon_1 a_1 + \cdots + \varepsilon_n a_n| \leq 1 \right) > \delta.$$

Please do NOT submit written solutions to the following exercises:

Exercise 1. Modify the argument we used to show that every $n$-uniform hypergraph with fewer than $2^{n-1}$ edges is 2-colourable so that it yields the stronger assertion that every $n$-uniform hypergraph with at most $2^{n-1}$ edges is 2-colourable.

Exercise 2. Show that the Ramsey number $R(4, k)$ satisfies $R(4, k) \geq c(k/ \log k)^2$ for some constant $c > 0$.

Exercise 3. Deduce from the theorem of Ajtai–Komlós–Szemerédi that $R(3, k) \leq Ck^2/ \log k$ for some constant $C$. 