Probabilistic methods in combinatorics

Homework assignment #2

Due date: Monday, May 13, 2019

Problem 1. Suppose that $v_1, \ldots, v_n$ are two-dimensional vectors whose coordinates are positive integers not exceeding $2^{n/2}/(10\sqrt{n})$. Prove that there are two disjoint nonempty sets $I, J \subseteq \{1, \ldots, n\}$ such that

$$\sum_{i \in I} v_i = \sum_{j \in J} v_j.$$

Problem 2. Prove that there is a constant C such that, with probability tending to one as $n \to \infty$, the largest number of edges in a bipartite subgraph of the random graph $G(n, 1/2)$ is at most $n^2/8 + Cn^{3/2}$.

Problem 3. Prove that there is some constant $\delta > 0$ such that the following holds for every $n \geqslant 2$. The unit sphere in $\mathbb{R}^n$ contains a set P of at least $(1+\delta)^n$ points such that the (Euclidean) distance between every pair of distinct points in P is at least one.

Problem 4. Prove that there exists a positive integer k_0 with the following property. For every $k \geqslant k_0$, there exists a $\{\text{red}, \text{blue}\}$-colouring of $\mathbb{Z}$ such that no k-term arithmetic progression with common difference less than 1.99^k is monochromatic.

Problem 5. A coloring c of the vertices of a graph G is *nonrepetitive* if there is no simple path $v_1 \ldots v_{2\ell}$ in G with $c(v_i) = c(v_{\ell+i})$ for each i. Prove that for every positive integer D there is a constant C such that every graph G with maximum degree at most D admits a nonrepetitive coloring with C colors.

Problem 6. Let G be a graph with m edges and let S be a subset of $V(G)$ selected uniformly at random. Prove that $e_G(S) = 0$ with probability at least $(3/4)^m$.

Please do NOT submit written solutions to the following exercises:

Exercise 1. Let X be a random variable taking nonnegative integer values. Prove that

$$\Pr(X = 0) \leqslant \frac{\mathrm{Var}[X]}{\mathbb{E}[X^2]}.$$

Exercise 2. Let P denote the probability that the random graph $G(n, 1/2)$ contains a Hamilton cycle (HC) and let Q denote the probability that a uniformly chosen random coloring of the edges of K_n with red and blue contains both a red HC and a blue HC. Is $Q \leqslant P^2$?