Probabilistic methods in combinatorics

Homework assignment #3

Due date: Monday, May 27, 2019

Problem 1. Let $\mathcal{PD}(K_4)$ denote the following graph property: $G \in \mathcal{PD}(K_4)$ if and only if G contains a collection of $\lfloor |V(G)|/8 \rfloor$ pairwise vertex-disjoint copies of K_4. Find a function $\theta \colon \mathbb{N} \to [0,1]$ such that

$$\lim_{n\to\infty} \Pr\left(G_{n,p(n)} \in \mathcal{PD}(K_4)\right) = \begin{cases} 1 & \text{if } p(n) \gg \theta(n), \\ 0 & \text{if } p(n) \ll \theta(n). \end{cases} \tag{1}$$

(The problem asks not only to determine θ but also to prove that this θ satisfies (1).)

Problem 2. For a graph G, let $\alpha_3(G)$ denote the largest size of a set $U \subseteq V(G)$ such that the subgraph $G[U]$ induced by U contains no triangles. Show that there are constants $c, C > 0$ such that

$$\Pr\left(c \log n \leqslant \alpha_3(G_{n,1/2}) \leqslant C \log n\right) = 1 - o(1).$$

Problem 3. Prove that, for every positive ε, there is an n_0 such that, for every $n > n_0$, there is an n-vertex graph that contains *every* graph on $\lfloor (2-\varepsilon)\log_2 n \rfloor$ vertices as an *induced* subgraph.

Problem 4. Show that moment bounds for tail probabilities are always better than Cramér–Chernoff bounds. More precisely, let X be a nonnegative random variable such that $\mathbb{E}[e^{\lambda X}] < \infty$ for every $\lambda \geqslant 0$ and let $t > 0$. Prove that

$$\inf_{q\in\mathbb{N}} \frac{\mathbb{E}\left[X^q\right]}{t^q} \leqslant \inf_{\lambda>0} \frac{\mathbb{E}\left[e^{\lambda X}\right]}{e^{\lambda t}},$$

where $\mathbb{N}$ is the set of natural numbers, that is, $\mathbb{N} = \{0, 1, 2, \dots\}$.

Problem 5. Show that there exists an n_0 such that the following holds. Let G be a graph with $n \geqslant n_0$ vertices and minimum degree $\delta(G) \geqslant (\log n)^2$. The vertex set of G may be partitioned into three sets V_1, V_2, and V_3 such that $\delta(G[V_i]) \geqslant 0.33 \cdot \delta(G)$ for every i.

Problem 6. Let G be a graph with $\chi(G) = 2000$. Let U be a subset of $V(G)$ selected uniformly at random and let $H = G[U]$ be the subgraph of G induced by U. Prove that

$$\Pr\left(\chi(H) \leqslant 900\right) \leqslant \frac{1}{10}.$$

Please do NOT submit written solutions to the following exercises:

Exercise 1. Let X be a square-integrable real-valued random variable and let m be a median of X, that is, m is a number such that both $X \leqslant m$ and $X \geqslant m$ hold with probability at least $1/2$. Show that

$$\left|\mathbb{E}[X] - m\right| \leqslant \sqrt{\operatorname{Var}(X)}.$$