Probabilistic methods in combinatorics

Homework assignment #4

Due date: Monday, June 10, 2019

Problem 1. Let $\mathcal{G}$ be a family of graphs with vertex set $[2n]$ such that the intersection of every pair of graphs in $\mathcal{G}$ contains a perfect matching. Prove that $|\mathcal{G}| \leqslant 2^{\binom{2n}{2}-n}$.

Problem 2. Let $\mathcal{F}_n$ denote the family of all triangle-free graphs with vertex set $[n]$. Prove that the sequence $\binom{n}{2}^{-1} \log |\mathcal{F}_n|$ is decreasing. Deduce that $|\mathcal{F}_n| \leqslant 7^{\binom{n}{2}/3}$ for all $n \geqslant 3$.

Problem 3. Let A be a finite set of finite sequences of elements of $[r]$. Suppose that no two distinct concatenations of sequences in A can produce the same string. Prove that

$$\sum_{a \in A} r^{-|a|} \leqslant 1,$$

where $|a|$ is the length of the sequence a.

Definition. The length of a longest common subsequence of two sequences x and y, denoted by $\mathrm{LCS}(x, y)$, is the largest integer ℓ such that there are $i_1 < \ldots < i_\ell$ and $j_1 < \ldots < j_\ell$ for which $x_{i_k} = y_{j_k}$ for all $k \in [\ell]$.

Problem 4. Let M be a fixed integer and suppose that $x = x(n)$ and $y = y(n)$ are two independent uniformly chosen $\{1, \ldots, M\}$-valued sequences of length n. Prove that there exists a constant $c = c(M)$ with $0 < c < 1$ such that

$$\lim_{n \to \infty} \frac{\mathbb{E}[\mathrm{LCS}(x, y)]}{n} = c.$$

Problem 5. Let $M \colon \mathbb{N} \to \mathbb{N}$ be arbitrary and suppose that $x = x(n)$ and $y = y(n)$ are two independent uniformly chosen $\{1, \ldots, M(n)\}$-valued sequences of length n. Prove that there exists an $\ell \colon \mathbb{N} \to \mathbb{N}$ such that for every $\omega \colon \mathbb{N} \to \mathbb{R}$ satisfying $\omega(n) \to \infty$,

$$\lim_{n \to \infty} \Pr\left(\left| \mathrm{LCS}(x, y) - \ell(n) \right| \leqslant \omega(n) \sqrt{\ell(n)} \right) = 1.$$