Probabilistic Methods in Combinatorics

Homework assignment #1

Due date: Sunday, November 23, 2014

Problem 1. Prove that the off-diagonal Ramsey numbers $R(4,k)$ satisfy

$$R(4,k) \geqslant \Omega\left(\frac{k^2}{(\log k)^2}\right).$$

Problem 2. Let G be a bipartite graph with 2^n vertices and suppose that each vertex of G is given a list of n colors. Assuming that $n \geqslant 2$, prove that there is a proper coloring of G that assigns to each vertex a color from its list.

Problem 3. Let $\mathcal{A}$ be a family of subsets of $\{1,\ldots,n\}$ such that there are no $A, B \in \mathcal{A}$ with $A \subseteq B$. Prove that

$$|\mathcal{A}| \leqslant \binom{n}{\lfloor n/2 \rfloor}$$

by considering a random permutation σ of $\{1,\ldots,n\}$ and computing the expectation of the random variable X defined by

$$X(\sigma) = \big|\big\{i \in [n] \colon \{\sigma(1),\ldots,\sigma(i)\} \in \mathcal{A}\big\}\big|.$$

Problem 4. Suppose that G is a graph with minimum degree at least two. Show that there exist three pairwise disjoint independent sets A, B, and C with

$$|A| + |B| + |C| \geqslant \sum_{v \in V(G)} \frac{3}{\deg_G(v) + 1}.$$

Problem 5. Let $\mathcal{F}$ be a finite collection of binary strings of finite lengths such that no member of $\mathcal{F}$ is a prefix of another. Let N_i denote the nuber of strings of length i in $\mathcal{F}$. Prove that

$$\sum_{i=1}^{\infty} \frac{N_i}{2^i} \leqslant 1.$$

Problem 6. Prove that an arbitrary set of ten points in the plane can be covered by a family of pairwise disjoint unit disks.