Probabilistic Methods in Combinatorics

Homework assignment #3

Due date: Sunday, January 4, 2015

Problem 1. Let G be a graph with m edges and let S be a subset of $V(G)$ selected uniformly at random. Prove that $e_G(S) = 0$ with probability at least $(3/4)^m$.

Problem 2. Let P denote the probability that the random graph $G(n, 1/2)$ contains a Hamilton cycle (HC) and let Q denote the probability that a uniformly chosen random coloring of the edges of K_n with red and blue contains both a red HC and a blue HC. Is $Q \leqslant P^2$?

Problem 3. A family of subsets $\mathcal{F}$ is called intersecting if $A_1 \cap A_2 \neq \emptyset$ for all $A_1, A_2 \in \mathcal{F}$. Let $\mathcal{F}_1, \ldots, \mathcal{F}_k$ be k intersecting families of subsets of $\{1, \ldots, n\}$. Prove that

$$\left| \bigcup_{i=1}^{k} \mathcal{F}_i \right| \leqslant 2^n - 2^{n-k}.$$

Problem 4. Show that there exists an n_0 such that the following holds. Let G be a graph with $n \geqslant n_0$ vertices and minimum degree $\delta(G) \geqslant (\log n)^2$. The vertex set of G may be partitioned into three sets V_1, V_2, and V_3 such that $\delta(G[V_i]) \geqslant 0.33 \cdot \delta(G)$ for every i.

Problem 5. Let G be a graph with $\chi(G) = 2000$. Let U be a subset of $V(G)$ selected uniformly at random and let $H = G[U]$ be the subgraph of G induced by U. Prove that

$$\Pr\left(\chi(H) \leqslant 900\right) \leqslant \frac{1}{10}.$$