Random graphs

Homework assignment #1

Problem 1. Prove that every nontrivial monotone graph property has a threshold in $G_{n,m}$ directly, that is, without appealing to the asymptotic equivalence between $G_{n,p}$ and $G_{n,m}$.

Problem 2. Show that for every $r \geqslant 3$ and $\varepsilon > 0$, there is a constant $C > 0$ such that the following holds. If $p \geqslant Cn^{-2/r}$, then $G_{n,p}$ a.a.s. contains a collection of at least $(1 - \varepsilon)n/r$ vertex-disjoint copies of K_r.

Problem 3. Show that for every $r \geqslant 3$, there is a constant $c > 0$ such that the following holds. If $p \leqslant cn^{-2/r}(\log n)^{1/\binom{r}{2}}$, then $G_{n,p}$ a.a.s. does not have a K_r-factor (a collection of n/r vertex-disjoint copies of K_r).

Remark. It is true, for every $r \geqslant 3$, that if $p \geqslant Cn^{-2/r}(\log n)^{1/\binom{r}{2}}$ and n is divisible by r, then $G_{n,p}$ a.a.s. contains a K_r-factor. This was proved by Johansson, Kahn, and Vu (2008).

Problem 4. Let k be a nonnegative integer, let $p = \frac{1}{n}(\log n + k \log\log n + C(n))$, and suppose that $G \sim G_{n,p}$. Show that if $C(n) \to -\infty$, then a.a.s. $\delta(G) \leqslant k$.

Problem 5. Suppose that $p = c/n$, where $c > 1$ is a constant. Show that a.a.s. $G_{n,p}$ is not planar.

Problem 6. Show that for every integer $k \geqslant 3$ and real $c > 0$, there exists $\theta > 0$ such that the following holds. If $G \sim G_{n,p}$ and $p \leqslant c/n$, then a.a.s. every subset A with $\delta(G[A]) \geqslant k$ has at least θn elements.

Remark. The *k-core* of a graph G is the largest subgraph H of G with $\delta(H) \geqslant k$. The previous problem implies that a.a.s. the k-core of $G_{n,p}$ is either empty or has $\Omega(n)$ vertices.