Random graphs
Homework assignment #1

Problem 1. Prove that every nontrivial monotone graph property has a threshold in $G_{n,m}$ directly, that is, without appealing to the asymptotic equivalence between $G_{n,p}$ and $G_{n,m}$.

Problem 2. Show that for every $r \geq 3$ and $\varepsilon > 0$, there is a constant $C > 0$ such that the following holds. If $p \geq Cn^{-2/r}$, then $G_{n,p}$ a.a.s. contains a collection of at least $(1 - \varepsilon)n/r$ vertex-disjoint copies of $K_r$.

Problem 3. Show that for every $r \geq 3$, there is a constant $c > 0$ such that the following holds. If $p \leq cn^{-2/r} (\log n)^{1/r}$, then $G_{n,p}$ a.a.s. does not have a $K_r$-factor (a collection of $n/r$ vertex-disjoint copies of $K_r$).

Remark. It is true, for every $r \geq 3$, that if $p \geq Cn^{-2/r} (\log n)^{1/r}$ and $n$ is divisible by $r$, then $G_{n,p}$ a.a.s. contains a $K_r$-factor. This was proved by Johansson, Kahn, and Vu (2008).

Problem 4. Let $k$ be a nonnegative integer, let $p = \frac{1}{n} (\log n + k \log \log n + C(n))$, and suppose that $G \sim G_{n,p}$. Show that if $C(n) \to -\infty$, then a.a.s. $\delta(G) \leq k$.

Problem 5. Suppose that $p = c/n$, where $c > 1$ is a constant. Show that a.a.s. $G_{n,p}$ is not planar.

Problem 6. Show that for every integer $k \geq 3$ and real $c > 0$, there exists $\theta > 0$ such that the following holds. If $G \sim G_{n,p}$ and $p \leq c/n$, then a.a.s. every subset $A$ with $\delta(G[A]) \geq k$ has at least $\theta n$ elements.

Remark. The $k$-core of a graph $G$ is the largest subgraph $H$ of $G$ with $\delta(H) \geq k$. The previous problem implies that a.a.s. the $k$-core of $G_{n,p}$ is either empty or has $\Omega(n)$ vertices.