Random graphs

Homework assignment #1

**Problem 1.** Prove that every nontrivial monotone graph property has a threshold in $G_{n,m}$ directly, that is, without appealing to the asymptotic equivalence between $G_{n,p}$ and $G_{n,m}$.

**Problem 2.** Show that for every $r \geq 3$ and $\varepsilon > 0$, there is a constant $C > 0$ such that the following holds. If $p \geq C n^{-2/r}$, then $G_{n,p}$ a.a.s. contains a collection of at least $(1 - \varepsilon)n/r$ vertex-disjoint copies of $K_r$.

**Problem 3.** Show that for every $r \geq 3$, there is a constant $c > 0$ such that the following holds. If $p \leq cn^{-2/r}(\log n)^{1/(\lfloor r/2 \rfloor)}$, then $G_{n,p}$ a.a.s. does not have a $K_r$-factor (a collection of $n/r$ vertex-disjoint copies of $K_r$).

**Remark.** It is true, for every $r \geq 3$, that if $p \geq C n^{-2/r}(\log n)^{1/(\lfloor r/2 \rfloor)}$ and $n$ is divisible by $r$, then $G_{n,p}$ a.a.s. contains a $K_r$-factor. This was proved by Johansson, Kaln, and Vu (2008).

**Problem 4.** Let $k$ be a nonnegative integer, let $p = \frac{1}{n}(\log n + k \log \log n + C(n))$, and suppose that $G \sim G_{n,p}$. Show that if $C(n) \to -\infty$, then a.a.s. $\delta(G) \leq k$.

**Problem 5.** Suppose that $p = c/n$, where $c > 1$ is a constant. Show that a.a.s. $G_{n,p}$ is not planar.

**Problem 6.** Show that for every integer $k \geq 3$ and real $c > 0$, there exists $\theta > 0$ such that the following holds. If $G \sim G_{n,p}$, then a.a.s. every subset $A$ with $\delta(G[A]) \geq k$ has at least $\theta n$ elements.

**Remark.** The $k$-core of a graph $G$ is the largest subgraph $H$ of $G$ with $\delta(H) \geq k$. The previous problem implies that a.a.s. the $k$-core of $G_{n,p}$ is either empty or has $\Omega(n)$ vertices.