Algebraic torus actions on Fukaya categories

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Overview

1. Motivation and the main result
2. Algebraic torus action on $\mathcal{F}(M)$
3. Group action property
4. Other corollaries and applications
Let \((M, \omega)\) be a closed symplectic manifold. Given closed 1-form \(\alpha\), define \(X_\alpha\) by \(\omega(\cdot, X_\alpha) = \alpha\), let \(\varphi_\alpha^t\) denote flow of \(X_\alpha\).

Given (nice) Lagrangians \(L, L' \subset M\), we have the family of Floer homology groups \(HF(L, \varphi_\alpha^t(L'))\) parametrized by \(t\).

More generally, given \(\nu \in H^1(M, \mathbb{R})\), let \(\varphi_\nu = \varphi_\alpha^1\) for some \(\alpha\) such that \(\nu = [\alpha]\). We obtain family

\[
\{HF(L, \varphi_\nu(L')) : \nu \in H^1(M, \mathbb{R})\}
\]
Motivation

Example

\[ M = \mathbb{R}^2 / \mathbb{Z}^2, \quad dx\, dy \]

\[ \partial_y = -\chi_{\downarrow x} \quad \Rightarrow \quad L = \mathbb{L} \quad \Rightarrow \quad \partial_x = \chi_{\downarrow y} \]

Then,

\[
HF(L, \varphi_v(L')) = \begin{cases} 
H^*(S^1), & \text{if } v \in \mathbb{Z} \times \mathbb{R} \\
0, & \text{otherwise}
\end{cases}
\]
Motivation

Example (cont’d)

\[ HF(L, \varphi_v(L')) = \begin{cases} 
H^*(S^1), & \text{if } v \in \mathbb{Z} \times \mathbb{R} \\
0, & \text{otherwise} 
\end{cases} \]

Restrict to \( \mathbb{R} \times \{0\} \), support is

\[ \cdots \]

Observe: Not an algebraic set, cannot be defined using polynomials of \( x, e^x \), etc.
Motivation

Extend by local systems:

Notation

Let $\Lambda = \mathbb{C}((T^\mathbb{R}))$, $\mathbb{G}_m = \Lambda^*$. Define $U_\Lambda := \text{val}_{T^{-1}}(0) = \{a + \text{higher powers of } T : a \in \mathbb{C}^*\} = \text{"the unitary group" } \subset \mathbb{G}_m$.

For any $\xi \in H^1(M, U_\Lambda)$, unitary local system, define $HF(L, (L', \xi|_{L'}))$. Observe, $\mathbb{G}_m \simeq \mathbb{R} \times U_\Lambda$, $T^r \xi \mapsto (r, \xi)$. Hence,

$$H^1(M, \mathbb{G}_m) \xrightarrow{\sim} H^1(M, \mathbb{R}) \times H^1(M, U_\Lambda)$$

$$z = (T^{v_1} \xi_1, T^{v_2} \xi_2, \ldots) \mapsto ((v_1, v_2, \ldots), (\xi_1, \xi_2, \ldots))$$

i.e. “$z = T^v \xi$”. Let $\varphi_z(L) := (\varphi_v(L), \xi|_L)$. We get a family

$$\{HF(L, \varphi_z(L')) : z \in H^1(M, \mathbb{G}_m)\}$$
Motivation

Remark

One expects to fit this family into an “analytic sheaf”, but not an algebraic one (as torus example has shown).

Question: Is it ever algebraic?
Main result

**Theorem 1**

Let \((M, \omega)\) be negatively monotone, integral, “strongly non-degenerate”, \(L, L'\) be tautologically unobstructed. Then, there exists an algebraic coherent sheaf (more precisely, a complex of such) over \(H^1(M, \mathbb{G}_m)\), whose restriction at \(z\) has cohomology \(HF(L, \varphi_z(L'))\).

**Remark**

Theorem 1 also holds for \(M\) Weinstein, \(L, L'\) compact, but requires other techniques.

**Corollary**

\[ \dim(HF(L, \varphi_z(L'))) \text{ is constant for } z \text{ in a non-empty Zariski open subset of } H^1(M, \mathbb{G}_m). \]
Main result

**Corollary**

Given $\alpha$ as before, $\dim(\text{HF}(L, \varphi^t_\alpha(L'))) \text{ is constant in } t$, with finitely many exceptions.

**Example**

$$\text{HF}(L, \varphi^t_\alpha(L')) = \begin{cases} H^*(S^1), & \text{at a single } t \in \mathbb{R} \\ 0, & \text{otherwise} \end{cases}$$

More sophisticated examples can be constructed on $M = \Sigma_2 \times \Sigma_2$, etc.
Assumptions

- $M$ is non-degenerate (i.e. satisfies generation criteria) $\Rightarrow$ technical assumption
- $\mathcal{F}(M)$ is generated by a set of **Bohr-Sommerfeld monotone** Lagrangians $\{L_i\}$

**B-S monotone** $\Rightarrow$ there are finitely many rigid holomorphic discs with fixed boundary conditions on $\{L_i\}$

Notation

$\mathcal{F}(M)$ denotes the Fukaya category with objects $\{L_i\}$. 
Main tool: algebraic torus action

Construct an action of $H^1(M, \mathbb{G}_m)$ on the Fukaya category, by quasi-functors.

- Quasi-functor $= A_\infty$-bimodule $=$ instead of telling $\varphi_z \lhd F(M)$, we tell $HF(L_i, \varphi_z(L_j))$ (c.f. quilted Floer homology)
- (Algebraic) action by quasi-functors $=$ (algebraic) family of bimodules

**Definition**

Let $\Phi|_z(L_i, L_j) = \Lambda \langle L_i \cap L_j \rangle$. Define $\mu^1(x) = \sum \pm T^E(u) z[\partial_h u] y$, where $u$ varies over

![Diagram](image.png)
Remark

The sums $\sum \pm T^E(u) z^{[\partial_h u]} y$ are finite due to Bohr-Sommerfeld condition, so $\Phi|_z$ is defined for all $z \in H^1(M, \mathbb{G}_m)$. 
Observe $\Lambda[z^{H_1(M)}] = \mathcal{O}(H^1(M, \mathbb{G}_m))$.

**Definition**

Define family $\Phi$ of bimodules by $\Phi(L_i, L_j) = \Lambda[z^{H_1(M)}] \langle L_i \cap L_j \rangle$ and $\mu^1(x) = \sum \pm T^E(u) z[\partial_h u] \cdot y$ as before. To define higher structure maps count

with weight $T^E(u) z[\partial_h u]$ as before.

$\Phi|_z$ can be obtained by evaluating at the specific $z \in H^1(M, \mathbb{G}_m)$. 
Lemma (Fukaya’s trick)

Let $z = T^v \xi$ be such that $v \in H^1(M, \mathbb{R})$ is close to 0. Then, $\Phi|_z$ corresponds to $\varphi_z$, i.e.

$$h_L \otimes F(M) \Phi|_z \simeq h_{\varphi_z(L)}$$

Terms and notation:

- $h_L =$ right Yoneda module of $L$, well-defined even if $L \not\in F(M)$
- $\otimes F(M) \Phi|_z =$ convolution with $\Phi|_z$. Should be thought as the action of the quasi-functor $\Phi|_z$ on $L$

Corollary

$$H^*(h_{L'} \otimes F(M) \Phi|_z \otimes F(M) h^L) \simeq H^*(h_{\varphi_z(L')} \otimes F(M) h^L) \simeq HF(L, \varphi_z(L'))$$ for $z = T^v \xi$ with small $v$. 

How geometric is $\Phi|_z$?

$h_{L'} \otimes \mathcal{F}(M) \Phi|_z \otimes \mathcal{F}(M) h^L$ can be obtained from $h_{L'} \otimes \mathcal{F}(M) \Phi \otimes \mathcal{F}(M) h^L$, by evaluating at $z$. Observe $h_{L'} \otimes \mathcal{F}(M) \Phi \otimes \mathcal{F}(M) h^L$

- is a complex of $\Lambda[z^{H_1(M)}] = \mathcal{O}(H^1(M, \mathbb{G}_m))$-modules
- is by construction algebraic
- has coherent cohomology (follows from abstract non-sense)

So, $h_{L'} \otimes \mathcal{F}(M) \Phi \otimes \mathcal{F}(M) h^L$ is our candidate for the algebraic sheaf mentioned in the theorem.

**Need:** Lemma above (hence, its corollary) to hold for all $z \in H^1(M, \mathbb{G}_m)$, i.e. $h_L \otimes \mathcal{F}(M) \Phi|_z \simeq h_{\varphi_z(L)}$. 
How geometric is $\Phi|_z$?

**Lemma**

If $\Phi|_{z_2} \otimes \mathcal{F}(M) \Phi|_{z_1} \simeq \Phi|_{z_1 z_2}$ hold for all $z_1, z_2$, then $h_L \otimes \mathcal{F}(M) \Phi|_z \simeq h_{\varphi_z(L)}$ for all $z$.

**Sketch of the proof.**

Assume $z = T^v$, $v \in H^1(M, \mathbb{R})$, fix $\alpha$ such that $v = [\alpha]$. Consider the isotopy $\varphi^t_\alpha(L)$, $t \in [0, 1]$.

By the lemma, for every $t$, there exists an $\epsilon_t > 0$ such that

$$h_{\varphi^t_\alpha(L)} \otimes \mathcal{F}(M) \Phi|_{T^sv} \simeq h_{\varphi^{t+s}_\alpha(L)}$$

for every $|s| < \epsilon_t$. Cover $[0, 1]$ by finitely many of $(t - \epsilon_t, t + \epsilon_t)$. Choose $0 = t_0 < t_1 < \cdots < t_k = 1$ such that two adjacent $t_i$ are in the same such interval. Then

$$h_{\varphi^1_\alpha(L)} \simeq h_L \otimes \mathcal{F}(M) \Phi|_{T^{t_1}v} \otimes \mathcal{F}(M) \Phi|_{T^{(t_2-t_1)v} \otimes \mathcal{F}(M)} \cdots \Phi|_{T^{(t_k-t_{k-1})v} \otimes \mathcal{F}(M)} \simeq$$

$$h_L \otimes \mathcal{F}(M) \Phi|_{T^{t_kv}} = h_L \otimes \mathcal{F}(M) \Phi|_z$$
Group action property

**Need:** \( \Phi|_{z_2} \otimes \mathcal{F}(M) \Phi|_{z_1} \simeq \Phi|_{z_1 z_2} \).

- Convolution \( \otimes \mathcal{F}(M) \) here can be thought as composition of quasi-functors.
- Hence, this condition is basically saying family \( \Phi \) is an action of \( H^1(M, \mathbb{G}_m) \) by quasi-functors.

Define a bimodule homomorphism

\[
F : \Phi|_{z_2} \otimes \mathcal{F}(M) \Phi|_{z_1} \to \Phi|_{z_1 z_2}
\]

by counting

with weight \( T^{E(u)} z_1^{[\partial_1 u]} z_2^{[\partial_2 u]} \) (c.f. Lekili-Lipyanskiy).
Abstract non-sense ⇒ $F$ is a quasi-isomorphism when $z_1, z_2 \in H^1(M, U_\Lambda)$

**Goal:** Show $F$ is a quasi-isomorphism everywhere

1. Compute the “deformation class” of $\Phi$ and $\text{cone}(F)$
2. $\Phi$, $\text{cone}(F)$ “follow” specific (Hochschild) cohomology classes
3. Hence, $\text{Hom}(\text{cone}(F), \text{cone}(F))$ carries a connection, also vanishes at $z_1, z_2 \in H^1(M, U_\Lambda) \subset H^1(M, \mathbb{G}_m)$
4. Abstract non-sense again ⇒ $\text{Hom}(\text{cone}(F), \text{cone}(F))$ is coherent

Therefore, $\text{Hom}(\text{cone}(F), \text{cone}(F))$ vanishes everywhere, i.e. $F$ is a quasi-isomorphism. This completes the proof of group action property
1. Construct an algebraic family $\Phi$ of quasi-functors of $\mathcal{F}(M)$

2. Fukaya’s trick $\Rightarrow \Phi\mid_z$ is geometric for small $z$ (i.e. acts like a symplectomorphism + unitary local system)

3. Write a transformation $F : \Phi\mid_{z_2} \otimes \mathcal{F}(M) \Phi\mid_{z_1} \rightarrow \Phi\mid_{z_1z_2}$, show that it is a quasi-isomorphism

4. Conclude $\Phi\mid_z$ is geometric for all $z$

5. Conclude $h_{L'} \otimes \mathcal{F}(M) \Phi \otimes \mathcal{F}(M) h^L$ has cohomology $HF(L, \varphi_z(L'))$ at $z$

This proves the main theorem. In other words, the groups $HF(L, \varphi_z(L'))$ fit into an algebraic sheaf.
Other corollaries, applications

**Corollary**

\[ \dim(\text{HF}(L, \varphi_z(L')) \) define an algebraic stratification of \( H^1(M, \mathbb{G}_m) \). \]

**Theorem 2**

“\text{The stabilizer}” \( \{ z : \varphi_z(L) \sim L \} \subset H^1(M, \mathbb{G}_m) \) form an algebraic subtorus of \( H^1(M, \mathbb{G}_m) \) with Lie algebra given by \( \ker(H^1(M, \Lambda) \to H^1(L, \Lambda)) \).

**Idea of the proof.**

The compositions

\[
\begin{align*}
\mu^2 : \text{HF}(\varphi_z(L), L) \otimes \text{HF}(L, \varphi_z(L)) &\to \text{HF}(L, L) \\
\mu^2 : \text{HF}(L, \varphi_z(L)) \otimes \text{HF}(\varphi_z(L), L) &\to \text{HF}(\varphi_z(L), \varphi_z(L))
\end{align*}
\]

also vary algebraically. Consider the locus of \( z \) where \( \mu^2 \)'s hit 1.

**Note:** The relation \( \sim \) is slightly weaker than a quasi-isomorphism (unless \( L \) is connected).
Other corollaries, applications

Corollary

If $\varphi^1_\alpha(L) \sim L$ (e.g. Hamiltonian isotopic), then $\alpha|_L = 0$.

A final application is to mirror symmetry (for this assume $M$ is Weinstein):

Theorem 3

Assume $\mathcal{W}(M)$ is equivalent to $D^b(\text{Coh}(X))$, where $X$ is a projective or affine variety, such that there exists an exact Lagrangian torus $L$ carried to (the structure sheaf of) a smooth point of $X$. Also assume $H^1(M, \Lambda) \to H^1(L, \Lambda)$ is surjective. Then, there exists an affine torus chart $G^b_1(L)^m \subset X$ around $x$ whose other points are mirror to Lagrangian tori isotopic to $L$. 
Thank you!