Algebraic torus actions on Fukaya categories

Yusuf Barış Kartal

Princeton University

February 5, 2021

Overview

1. Motivation and the main result
2. Algebraic torus action on $\mathcal{F}(M)$
3. Group action property
4. Other corollaries and applications

Motivation

Let (M, ω) be a closed symplectic manifold. Given closed 1-form α, define X_α by $\omega(\cdot, X_\alpha) = \alpha$, let φ_α^t denote flow of X_α.

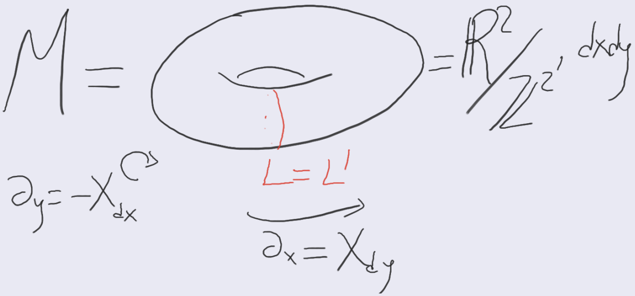
Given (nice) Lagrangians $L, L' \subset M$, we have the family of Floer homology groups $HF(L, \varphi_\alpha^t(L'))$ parametrized by t.

More generally, given $v \in H^1(M, \mathbb{R})$, let $\varphi_v = \varphi_\alpha^1$ for some α such that $v = [\alpha]$. We obtain family

$$\{HF(L, \varphi_v(L')) : v \in H^1(M, \mathbb{R})\}$$

Motivation

Example



Then,

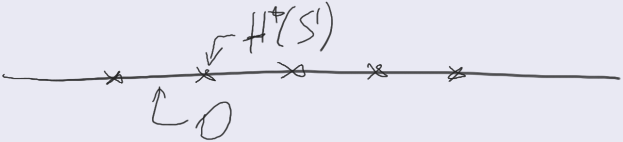
$$HF(L, \varphi_v(L')) = \begin{cases} H^*(S^1), & \text{if } v \in \mathbb{Z} \times \mathbb{R} \\ 0, & \text{otherwise} \end{cases}$$

Motivation

Example (cont'd)

$$HF(L, \varphi_v(L')) = \begin{cases} H^*(S^1), & \text{if } v \in \mathbb{Z} \times \mathbb{R} \\ 0, & \text{otherwise} \end{cases}$$

Restrict to $\mathbb{R} \times \{0\}$, support is



Observe: Not an algebraic set, cannot be defined using polynomials of x, e^x, etc.

Motivation

Extend by local systems:

Notation

Let $\Lambda = \mathbb{C}((T^{\mathbb{R}}))$, $\mathbb{G}_m = \Lambda^*$. Define $U_\Lambda := val_T^{-1}(0) = \{a + \text{higher powers of } T : a \in \mathbb{C}^*\} =$ "the unitary group" $\subset \mathbb{G}_m$.

For any $\xi \in H^1(M, U_\Lambda)$, unitary local system, define $HF(L, (L', \xi|_{L'}))$. Observe, $\mathbb{G}_m \cong \mathbb{R} \times U_\Lambda$, $T^r\xi \mapsto (r, \xi)$. Hence,

$$H^1(M, \mathbb{G}_m) \xrightarrow{\cong} H^1(M, \mathbb{R}) \times H^1(M, U_\Lambda)$$
$$z = (T^{v_1}\xi_1, T^{v_2}\xi_2, \dots) \mapsto ((v_1, v_2, \dots), (\xi_1, \xi_2, \dots))$$

i.e. "$z = T^v\xi$". Let $\varphi_z(L) := (\varphi_v(L), \xi|_L)$. We get a family

$$\{HF(L, \varphi_z(L')) : z \in H^1(M, \mathbb{G}_m)\}$$

Motivation

Remark

One expects to fit this family into an "analytic sheaf", but not an algebraic one (as torus example has shown).

Question: Is it ever algebraic?

Main result

Theorem 1

Let (M, ω) be negatively monotone, integral, "strongly non-degenerate", L, L' be tautologically unobstructed. Then, there exists an algebraic coherent sheaf (more precisely, a complex of such) over $H^1(M, \mathbb{G}_m)$, whose restriction at z has cohomology $HF(L, \varphi_z(L'))$.

Remark

Theorem 1 also holds for M Weinstein, L, L' compact, but requires other techniques.

Corollary

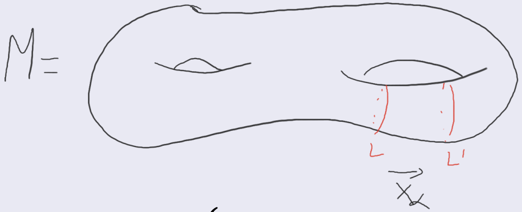
$dim(HF(L, \varphi_z(L')))$ is constant for z in a non-empty Zariski open subset of $H^1(M, \mathbb{G}_m)$.

Main result

Corollary

Given α as before, $\dim(HF(L, \varphi_\alpha^t(L')))$ is constant in t, with finitely many exceptions.

Example



$$HF(L, \varphi_\alpha^t(L')) = \begin{cases} H^*(S^1), & \text{at a single } t \in \mathbb{R} \\ 0 & \text{otherwise} \end{cases}$$

More sophisticated examples can be constructed on $M = \Sigma_2 \times \Sigma_2$, etc.

Assumptions

- M is non-degenerate (i.e. satisfies generation criteria) $\Rightarrow$ technical assumption
- $\mathcal{F}(M)$ is generated by a set of **Bohr-Sommerfeld monotone** Lagrangians $\{L_i\}$

B-S monotone $\Rightarrow$ there are finitely many rigid holomorphic discs with fixed boundary conditions on $\{L_i\}$

Notation

$\mathcal{F}(M)$ denotes the Fukaya category with objects $\{L_i\}$.

Main tool: algebraic torus action

Construct an action of $H^1(M, \mathbb{G}_m)$ on the Fukaya category, by quasi-functors.

- Quasi-functor=A_∞-bimodule=instead of telling $\varphi_z \curvearrowright \mathcal{F}(M)$, we tell $HF(L_i, \varphi_z(L_j))$ (c.f. quilted Floer homology)
- (Algebraic) action by quasi-functors= (algebraic) family of bimodules

Definition

Let $\Phi|_z(L_i, L_j) = \Lambda\langle L_i \cap L_j\rangle$. Define $\mu^1(x) = \sum \pm T^{E(u)} z^{[\partial_h u]}.y$, where u varies over

Main tool: algebraic torus action

Remark

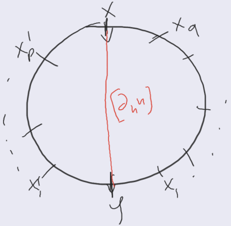
The sums $\sum \pm T^{E(u)} z^{[\partial_h u]}.y$ are finite due to Bohr-Sommerfeld condition, so $\Phi|_z$ is defined for all $z \in H^1(M, \mathbb{G}_m)$.

Main tool: algebraic torus action

Observe $\Lambda[z^{H_1(M)}] = \mathcal{O}(H^1(M, \mathbb{G}_m))$.

Definition

Define family Φ of bimodules by $\Phi(L_i, L_j) = \Lambda[z^{H_1(M)}]\langle L_i \cap L_j \rangle$ and $\mu^1(x) = \sum \pm T^{E(u)} z^{[\partial_h u]}.y$ as before. To define higher structure maps count



with weight $T^{E(u)} z^{[\partial_h u]}$ as before.

$\Phi|_z$ can be obtained by evaluating at the specific $z \in H^1(M, \mathbb{G}_m)$.

How geometric is $\Phi|_z$?

Lemma (Fukaya's trick)

Let $z = T^v\xi$ be such that $v \in H^1(M,\mathbb{R})$ is close to 0. Then, $\Phi|_z$ corresponds to φ_z, i.e.

$$h_L \otimes_{\mathcal{F}(M)} \Phi|_z \simeq h_{\varphi_z(L)}$$

Terms and notation:

- h_L=right Yoneda module of L, well-defined even if $L \notin \mathcal{F}(M)$
- $\otimes_{\mathcal{F}(M)}\Phi|_z$=convolution with $\Phi|_z$. Should be thought as the action of the quasi-functor $\Phi|_z$ on L

Corollary

$H^*(h_{L'} \otimes_{\mathcal{F}(M)} \Phi|_z \otimes_{\mathcal{F}(M)} h^L) \cong H^*(h_{\varphi_z(L')} \otimes_{\mathcal{F}(M)} h^L) \cong HF(L, \varphi_z(L'))$ for $z = T^v\xi$ with small v.

How geometric is $\Phi|_z$?

$h_{L'} \otimes_{\mathcal{F}(M)} \Phi|_z \otimes_{\mathcal{F}(M)} h^L$ can be obtained from $h_{L'} \otimes_{\mathcal{F}(M)} \Phi \otimes_{\mathcal{F}(M)} h^L$, by evaluating at z. Observe $h_{L'} \otimes_{\mathcal{F}(M)} \Phi \otimes_{\mathcal{F}(M)} h^L$

- is a complex of $\Lambda[z^{H_1(M)}] = \mathcal{O}(H^1(M, \mathbb{G}_m))$-modules
- is by construction algebraic
- has coherent cohomology (follows from abstract non-sense)

So, $h_{L'} \otimes_{\mathcal{F}(M)} \Phi \otimes_{\mathcal{F}(M)} h^L$ is our candidate for the algebraic sheaf mentioned in the theorem.

Need: Lemma above (hence, its corollary) to hold for all $z \in H^1(M, \mathbb{G}_m)$, i.e. $h_L \otimes_{\mathcal{F}(M)} \Phi|_z \simeq h_{\varphi_z(L)}$.

How geometric is $\Phi|_z$?

Lemma

If $\Phi|_{z_2} \otimes_{\mathcal{F}(M)} \Phi|_{z_1} \simeq \Phi|_{z_1 z_2}$ *hold for all* z_1, z_2*, then* $h_L \otimes_{\mathcal{F}(M)} \Phi|_z \simeq h_{\varphi_z(L)}$ *for all* z*.*

Sketch of the proof.

Assume $z = T^v, v \in H^1(M, \mathbb{R})$, fix α such that $v = [\alpha]$. Consider the isotopy $\varphi_\alpha^t(L), t \in [0,1]$.

By the lemma, for every t, there exists an $\epsilon_t > 0$ such that $h_{\varphi_\alpha^t(L)} \otimes_{\mathcal{F}(M)} \Phi|_{T^{sv}} \simeq h_{\varphi_\alpha^{t+s}(L)}$, for every $|s| < \epsilon_t$. Cover $[0,1]$ by finitely many of $(t - \epsilon_t, t + \epsilon_t)$. Choose $0 = t_0 < t_1 < \cdots < t_k = 1$ such that two adjacent t_i are in the same such interval. Then

$$h_{\varphi_\alpha^1(L)} \simeq h_L \otimes_{\mathcal{F}(M)} \Phi|_{T^{t_1 v}} \otimes_{\mathcal{F}(M)} \Phi|_{T^{(t_2 - t_1)v}} \otimes_{\mathcal{F}(M)} \dots \Phi|_{T^{(t_k - t_{k-1})v}} \simeq h_L \otimes_{\mathcal{F}(M)} \Phi|_{T^{t_k v}} = h_L \otimes_{\mathcal{F}(M)} \Phi|_z$$

□

Group action property

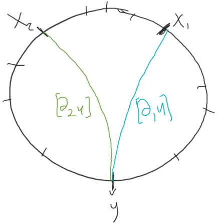
Need: $\Phi|_{z_2} \otimes_{\mathcal{F}(M)} \Phi|_{z_1} \simeq \Phi|_{z_1 z_2}$.

- Convolution $\otimes_{\mathcal{F}(M)}$ here can be thought as composition of quasi-functors.
- Hence, this condition is basically saying family Φ is an action of $H^1(M, \mathbb{G}_m)$ by quasi-functors.

Define a bimodule homomorphism

$$F : \Phi|_{z_2} \otimes_{\mathcal{F}(M)} \Phi|_{z_1} \to \Phi|_{z_1 z_2}$$

by counting



with weight $T^{E(u)} z_1^{[\partial_1 u]} z_2^{[\partial_2 u]}$ (c.f. Lekili-Lipyanskiy).

Group action property

Abstract non-sense $\Rightarrow$ F is a quasi-isomorphism when $z_1, z_2 \in H^1(M, U_\Lambda)$
Goal: Show F is a quasi-isomorphism everywhere

1. Compute the "deformation class" of Φ and $cone(F)$
2. Φ, $cone(F)$ "follow" specific (Hochschild) cohomology classes
3. Hence, $Hom(cone(F), cone(F))$ carries a connection, also vanishes at $z_1, z_2 \in H^1(M, U_\Lambda) \subset H^1(M, \mathbb{G}_m)$
4. Abstract non-sense again $\Rightarrow$ $Hom(cone(F), cone(F))$ is coherent

Therefore, $Hom(cone(F), cone(F))$ vanishes everywhere, i.e. F is a quasi-isomorphism. This completes the proof of group action property

Summary

1. Construct an algebraic family Φ of quasi-functors of $\mathcal{F}(M)$
2. Fukaya's trick $\Rightarrow \Phi|_z$ is geometric for small z (i.e. acts like a symplectomorphism+unitary local system)
3. Write a transformation $F : \Phi|_{z_2} \otimes_{\mathcal{F}(M)} \Phi|_{z_1} \to \Phi|_{z_1 z_2}$, show that it is a quasi-isomorphism
4. Conclude $\Phi|_z$ is geometric for all z
5. Conclude $h_{L'} \otimes_{\mathcal{F}(M)} \Phi \otimes_{\mathcal{F}(M)} h^L$ has cohomology $HF(L, \varphi_z(L'))$ at z

This proves the main theorem. In other words, the groups $HF(L, \varphi_z(L'))$ fit into an algebraic sheaf.

Other corollaries, applications

Corollary

dim$(HF(L,\varphi_z(L')))$ *define an algebraic stratification of* $H^1(M,\mathbb{G}_m)$.

Theorem 2

"The stabilizer" $\{z : \varphi_z(L) \sim L\} \subset H^1(M,\mathbb{G}_m)$ *form an algebraic subtorus of* $H^1(M,\mathbb{G}_m)$ *with Lie algebra given by* $\ker(H^1(M,\Lambda) \to H^1(L,\Lambda))$.

Idea of the proof.

The compositions

$$\mu^2 : HF(\varphi_z(L),L) \otimes HF(L,\varphi_z(L)) \to HF(L,L)$$

$$\mu^2 : HF(L,\varphi_z(L)) \otimes HF(\varphi_z(L),L) \to HF(\varphi_z(L),\varphi_z(L))$$

also vary algebraically. Consider the locus of z where μ^2's hit 1. □

Note: The relation $\sim$ is slightly weaker than a quasi-isomorphism (unless L is connected).

Other corollaries, applications

Corollary

If $\varphi_\alpha^1(L) \sim L$ (e.g. Hamiltonian isotopic), then $\alpha|_L = 0$.

A final application is to mirror symmetry (for this assume M is Weinstein):

Theorem 3

Assume $\mathcal{W}(M)$ is equivalent to $D^b(Coh(X))$, where X is a projective or affine variety, such that there exists an exact Lagrangian torus L carried to (the structure sheaf of) a smooth point of X. Also assume $H^1(M,\Lambda) \to H^1(L,\Lambda)$ is surjective. Then, there exists an affine torus chart $\mathbb{G}_m^{b_1(L)} \subset X$ around x whose other points are mirror to Lagrangian tori isotopic to L.

Thank you!