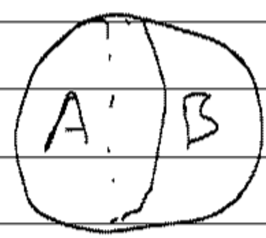
Nondisplaceable Lag links in 4-manifolds.

Note Title 2/12/2021

Eliashberg philosophy:
In symp topo, if something is not obstructed by J-holo curves then it should be flexible.

Example:



L

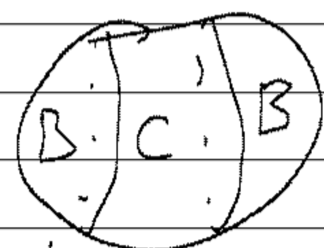
$(S^2, \omega) \supseteq L$

L is non-displaceable iff $A = B$

① use area reasoning to show that

② $HF(L) \neq 0$ if $A = B$

Q: what if

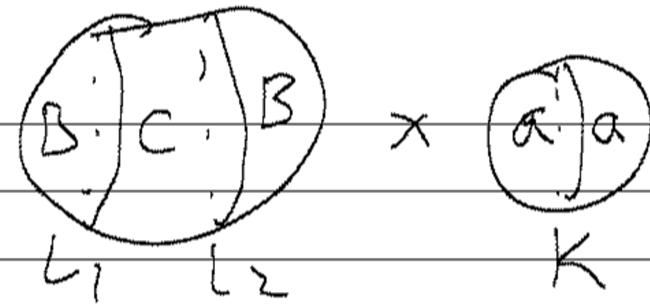


L_1 L_2

$L = L_1 \cup L_2$?

① ⇒ $C \leq 2B$ L is nondisplaceable

$C \leq B$ L_1 is nondisplaceable from L

Q: what if 

$\mathcal{L}_i = L_i \times K$ is $\mathcal{L} = \mathcal{L}_1 \cup \mathcal{L}_2$ non-displaceable when?
say $C \lesssim B$.

Prop (Polterovich)
$\mathcal{L}$ is displaceable when $2a > 2B + C$

Thm (M-Smith)
$\mathcal{L}_1$ is non-displaceable from $\mathcal{L}$ when $A + C \lesssim B$

Key idea: make use of J-holo annulus bded between $\mathcal{L}_1$ & $\mathcal{L}_2$

Holo annuli appear in Heegaard Floer

$M \supseteq \underline{L} = \{L_1 \dots L_p\}$

$\underline{K} = \{K_1, \dots K_p\}$

& $L_1 \times \dots L_p \subset M^p$

$\downarrow \qquad \downarrow$

$\underline{\mathrm{Sym}(\underline{L})} \subset \mathrm{Sym}^p(M) = M^p/S_p$

$HF(\mathrm{Sym}(\underline{L}), \mathrm{Sym}(\underline{K})) \leftarrow$ captures more holo curves bounded between $\cup L_i, \cup K_i$

(1) how ~~is it~~ is it related to our problem

(2) how ~~are~~ do these holo curves enter the story

A1: If L_0 is displaceable from L, say by ψ

then $\quad Sym(L) \cap Sym(\psi(L)) = \phi$

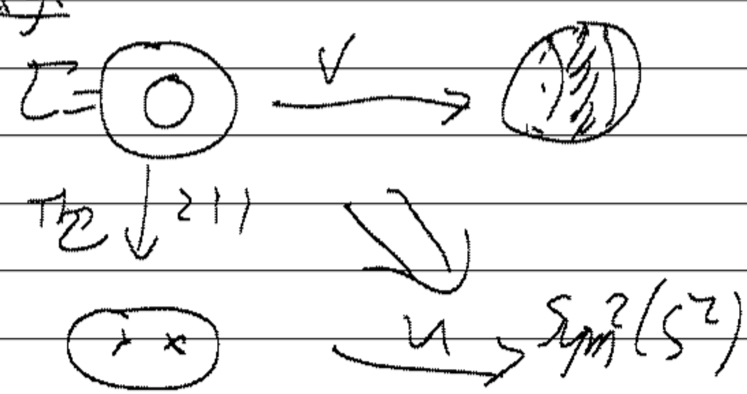
if $\boxed{Sym(L)}$ nondisplaceable, then so is L

A2: Tautological correspondence (Donaldson, Ozsvath-Szabo, Costello, etc..)

$$\begin{array}{ccc} \Sigma & \xrightarrow{\ v\ } & (M, \underline{L}) \quad \underline{L} = \bigcup_{i=1}^{p} L_i \\ {\scriptstyle p:1}\ \big\downarrow \pi_\Sigma & & \\ \cdot z & \xrightarrow{\ u\ } & (Sym^p(M), Sym(\underline{L})) \end{array}$$

$u(z) := v(\pi_\Sigma^{-1}(z))$

e.g.



Rest of the talk:

Prove the theorem using this idea & using FOOO, Cho-Poddar

M compact symp mfd/orbifold

$L \subset (M, \omega)$ torus $\left\{ \begin{array}{l} \text{bounds no Maslov} \leq 0 \text{ disks} \\ \text{away from orbifold loci} \end{array} \right.$

$H_1(L) = q_1\mathbb{Z} \oplus \cdots \oplus q_n\mathbb{Z}$

$$W_L(x_1, \dots x_n) = \sum_A \#\mathcal{M}_A \, T^{\omega \cdot A} x^{\partial A}$$

$\partial A = \sum a_i q_i, \quad x^{\partial A} = \prod x_i^{a_i}$

Thm (FOOO)

If W admits a crit pt $x \in (\Lambda_0 \setminus \Lambda_+)^n$

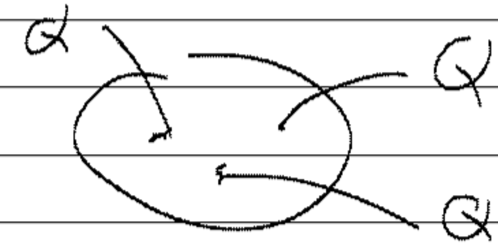
then $HF(L, x) \cong H(L)$

and L non-displaceable

Bulk-deformation:

Q cycle $\subseteq M$, $b \in \Lambda_+$

$$W^{bQ}(x_1, \dots, x_n) = \sum_A \left(\sum_{k=0}^{\infty} \left(\# \mathcal{M}_{A,kQ} \frac{b^k}{k!} \right) T^{\omega A} x^{\partial A} \right)$$



When M is an orbifold

$Q \subseteq$ orbifold twisted sectors.

e.g. $M = X/G$, $M_{(g)} \stackrel{\text{def}}{=} \underline{X^g / C(g)}$ for $g \in G$

$$M_e = M \Longleftarrow$$

inertia orbifold $\coprod_{(g)} M_{(g)}$ (g): conj class

$M_{(e)} = M$

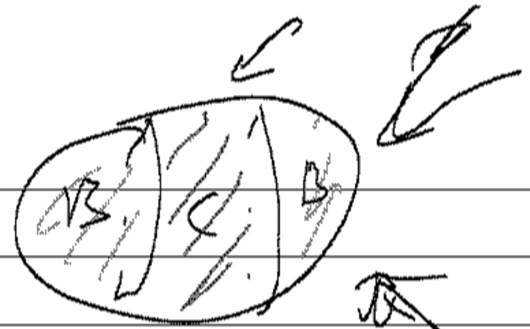
$M_{(g)} \neq M$, $g \neq e$

Σ

$z \in C$

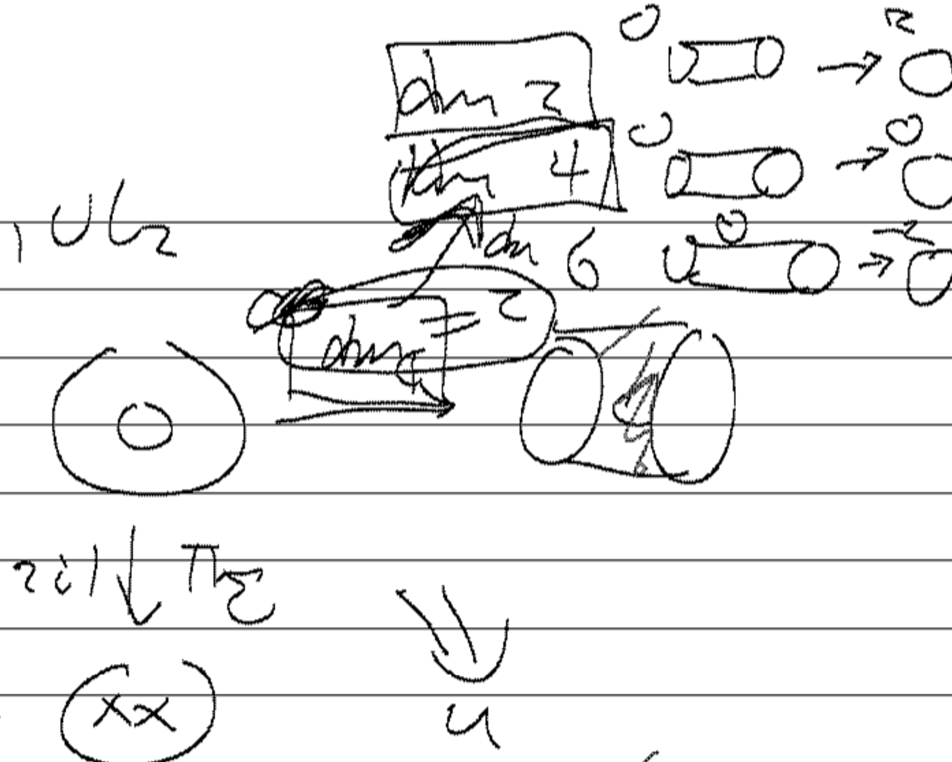
$ev_u(z) \in \coprod_{(g)} M_{(g)}$

e.g.



$L_1 \quad L_2$

$\underline{L} = L_1 \cup L_2$



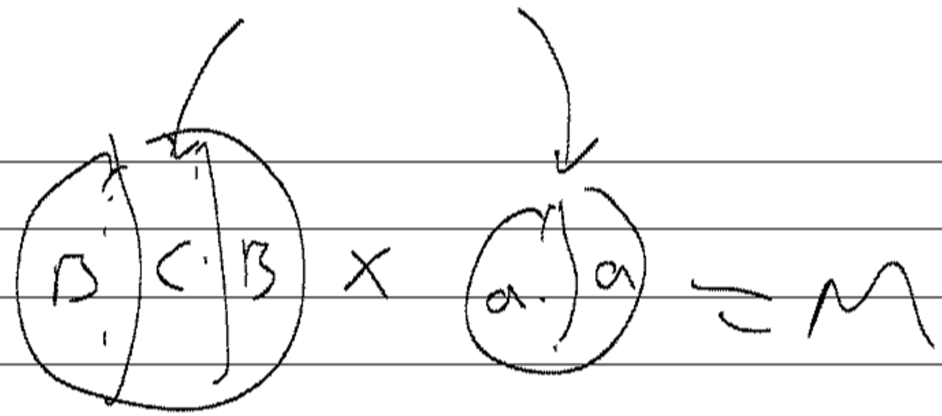
$M = \mathrm{Sym}^2(X) = X \times X / \mathbb{Z}_2$

$M_e = \mathrm{Sym}^2(X)$

$\underline{Q} = M_{(\neq e)} = \Delta \subseteq X \times X$

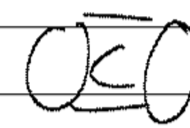
$\mathrm{Sym}(\underline{L}) \qquad W^{bQ}_{\mathrm{Sym}(\underline{L})} = x_1 T^B + x_2 T^B + \frac{b^2}{2} x_1^{-1} x_2^{-1} T^C$

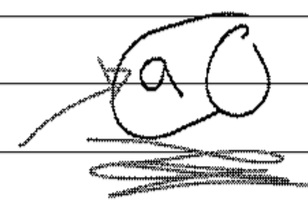
if $C < B$, then $\exists b$ s.t. W^{bQ} + h.o.t. has crit pt in $(\Lambda_0 \setminus \Lambda_+)^2$



★ When a is large, $\underline{L}$ displaceable

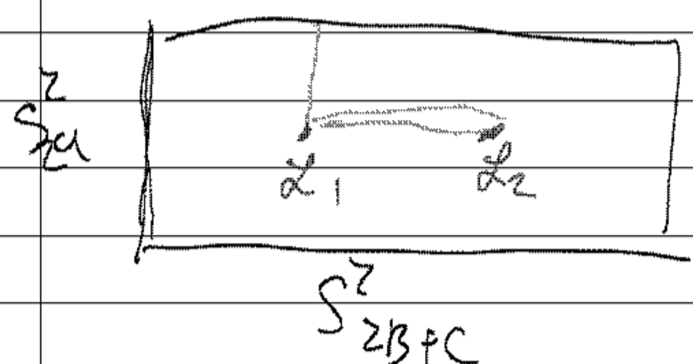
Annuli we want $\quad v = (v_1, v_2) : ◎ \longrightarrow M$

v_1 deg 1 surject onto 

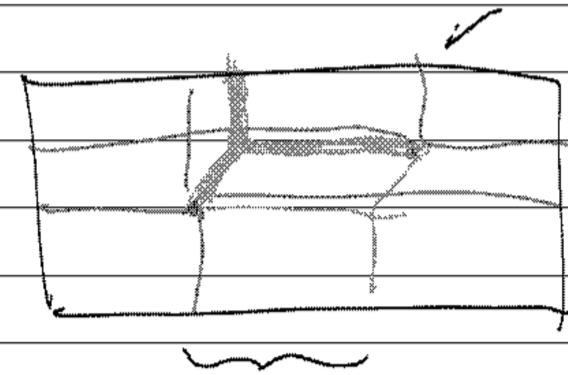
v_2 deg 1 surject onto 

Those ~~move one~~ annuli ~~ha~~ lifts to "Maslov 0" disk in $(Sym^2(S^2 \times S^2), \boxed{Sym(\underline{L})})$

Fukaya trick to move $Sym(\underline{L})$ away from "walls" (i.e. $Sym(\underline{L})$ no longer bound "Maslov 0" disk)



after Fuk $\xrightarrow{\text{trick}}$



C^2 small

Φ : diff of $S^2 \times S^2$ s.t.

$\Phi(L_1)$ and $\Phi(L_2)$ are torus fibers
w/ irrational slope

Basic complex analysis
$\Rightarrow \nexists$ holo annulus w/ ∂ on $\Phi(L_1)$
non-const Maslov 0 and $\Phi(L_2)$

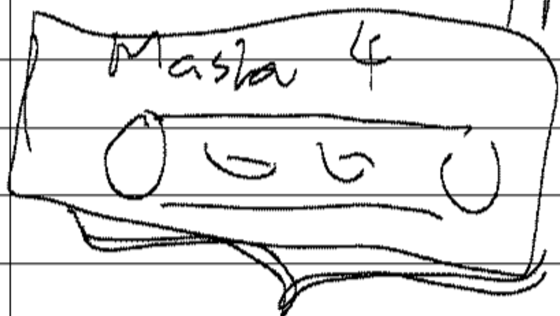
J_{std} - holo curve w/ ∂ on $\Phi(L_1)$ & $\Phi(L_2)$ $\leftarrow$

$\Leftrightarrow$ $\Phi^* J_{std}$ holo-curve w/ ∂ on L_1 & L_2 $\leftarrow$

Higher dimension

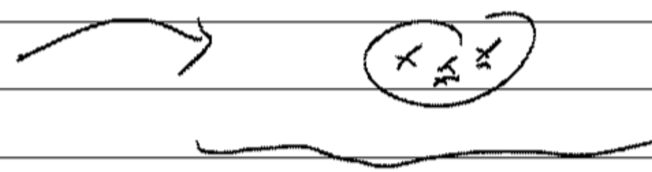
What could happen

Maslov 4



$\subseteq M$

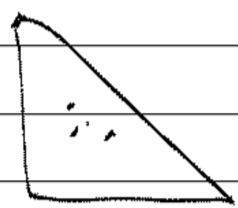
Maslov 0



$\mathrm{Sym}^k(M)$

$\dim(M) > 4$

$\notin M$



// edges

small distance

L is nondisplaceable from $\cup L_i$

∂L_i