Intrinsic Mirror Symmetry and Categorical Crepant resolutions
work in progress (based on earlier work with S. Ganatra)

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Algebro-geometric terminology

**Definition**

A *pair* \((M, D)\) will consist of a smooth projective variety (over \(\mathbb{C}\)) with \(D = \bigcup_i D_i\) a simple normal crossings divisor.

Being simple normal crossings means that each component \(D_i\) is smooth and all of the intersections \(D_i := \cap_{i \in I} D_i\) are transverse.

**Definition**

A *positive pair* \((M, D)\) is a pair such that \(D\) supports an ample line bundle, i.e. there is some ample \(L\) such that

\[
L \cong \mathcal{O}(\sum_i \kappa_i D_i)
\]

where \(\kappa_i\) are positive.
For positive pairs, the complement $X$ is an affine variety. We will be thinking of these affine varieties $X$ as exact symplectic manifolds.

Equip $M$ with a Kähler form $\omega_L$ associated to (a positive Hermitian metric $|| \cdot ||$ on) $L$ and restrict this form to $X$. $(\omega_L)_X$ has a natural primitive $\theta_L$ ($=-d^c h$ where $h$ is the Kähler potential).

The tuple $(X, \omega_L, \theta_L)$ equips $X$ with the structure of a (finite-type) convex symplectic manifold. So we can attach several “wrapped Floer theoretic” invariants

- $SH^*(X)$ (Cieliebak-Floer-Hofer, Viterbo)
- $\mathcal{W}(X)$ (Abouzaid-Seidel)

\[ h = -\log ||\beta|| \]

\[ \text{Wrapped Floer Coh.} \]
Log CY varieties

Definition

A pair \((M, D)\) is called a Calabi-Yau pair if \(D\) is an anti-canonical divisor. The complement \(X := M \setminus D\) is called a log Calabi-Yau variety.

This talk will be concerned with wrapped Floer invariants on affine log Calabi-Yau varieties.
Examples

- The most basic example of an affine log Calabi-Yau variety is $(\mathbb{C}^*)^n = \mathbb{C}P^n \setminus D$ where $D$ is the union of coordinate hyperplanes. The volume form is given in standard local coordinates by

  \[ \Omega := \frac{dz_1}{z_1} \wedge \cdots \wedge \frac{dz_n}{z_n} \]  

- A surprising rich class of examples is given by (affine) cluster varieties —roughly speaking varieties obtain by gluing copies of $(\mathbb{C}^*)^n$ according to specific rules which preserve this holomorphic volume form.
Kontsevich’s homological mirror symmetry (HMS) conjecture in this context predicts that in “nice” cases there is a mirror log Calabi-Yau (not necessarily affine) $Y$ such that:

$$\text{Perf}(\mathcal{W}(X)) \cong D^b\text{Coh}(Y)$$

(2)

$\text{Perf}(\mathcal{W}(X))$ is the (split-closed) derived wrapped Fukaya category of $X$ and $D^b\text{Coh}(Y)$ is the derived category of bounded coherent sheaves on $Y$.

Remarks

- One expects HMS to always hold in the above form when $\dim(X) \leq 3$. Many cases proven in $\dim(X) = 2$ (Pascaleff, Keating, Hacking-Keating).

- HMS has many more concrete consequences, for example it implies that the symplectic cohomology of $X$ can be computed in terms of sheaf cohomology groups on $Y$. 
Semi-affineness

It is a general expectation is that the mirror space $Y$ to an affine log Calabi-Yau $X$ should be semi-affine. This means that the canonical map: $\alpha : Y \twoheadrightarrow \text{Spec}(\Gamma(\mathcal{O}_Y))$ is proper. Semi-affineness has a number of interesting consequences, the two most important for us being:

- The ring of functions $\Gamma(\mathcal{O}_Y)$ is a finitely generated $k$-algebra.
- For any $E_0, E_1 \in D^b\text{Coh}(Y)$, $\text{RHom}^\ast_Y(E_0, E_1)$ is a finitely generated module over $\Gamma(\mathcal{O}_Y)$. 
The expectation that mirrors to $Y$ are semi-affine can be turned into the following precise theorem purely on $X$:

**Theorem (P, in progress)**

*For any affine log Calabi-Yau variety $X$:

1. The degree zero symplectic cohomology $\text{SH}^0(X)$ is finitely generated and is a filtered deformation of a certain algebra, $\mathcal{SR}(\Delta(D))$, defined combinatorially in terms of the compactifying divisor $D$.

2. For any $L_0, L_1$, the wrapped Floer groups $\text{WF}^*(L_0, L_1)$ are finitely generated modules over $\text{SH}^0(X)$.

As we will see later, this theorem together with some homological algebra leads to a criterion for HMS to hold “birationally.”
The combinatorial ring

Assume for simplicity that all strata $D_i$ are connected.

- For a vector $v = (v_i)$ in $(\mathbb{Z}_{\geq 0})^k$, we define the support of $v$, $|v|$ to be the set of $i \in \{1, \cdots, k\}$ such that $v_i \neq 0$. We let $B(M, D) \subseteq (\mathbb{Z}_{\geq 0})^k$ to be the set of vectors $v$ such that $D_{|v|} \neq \emptyset$.

- Let $A$ denote the vector space:

$$A := \bigoplus_{v \in B(M, D)} k \cdot \theta_v. \quad (3)$$

- We can equip $A$ with a ring structure

$$\theta_{v_1} \ast \theta_{v_2} = \begin{cases} 
\theta_{v_1 + v_2} & D_{|v_1 + v_2|} \neq \emptyset \\
0 & \text{otherwise}
\end{cases} \quad (4)$$

The ring $\mathcal{SR}(\Delta(D)) := (A, \ast)$.  

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In a recent paper Gross-Siebert have defined a (degree zero) “logarithmic quantum-cohomology” \( (\mathcal{A}, \star_{GS}) \), which is defined on the vector space \( \mathcal{A} \) and is a deformation of \( SR(\Delta(D)) \).

The deformation is given given by counting genus zero curves with tangencies (“punctured GW invariants”). It can often be computed essentially combinatorially using methods of tropical geometry (c.f. work of T. Mandel).

The Main Theorem suggests:

Conjecture

There is an isomorphism of rings \((\mathcal{A}, \star_{GS}) \cong SH^0(X, k)\).

Our proof of the Main Theorem is modelled on the usual PSS isomorphism between quantum cohomology and Floer cohomology and is hopefully a good first step in establishing a ring isomorphism.
Illustration of Theorem

- Let $\bar{M} = \mathbb{CP}^1 \times \mathbb{CP}^1$ and $\bar{D}$ be the toric divisors.
- Take $M$ to be the blowup of $\bar{M}$ at $n$ points in $\mathbb{CP}^1 \times \mathbb{CP}^1$. Let $D$ be the proper transform of $\bar{D}$.

$$SR(D) \cong \mathbb{K}[\theta_0, \theta_1, \theta_2, \theta_3]$$

$$(\theta_0 \cdot \theta_2 = 0, \theta_1 \theta_3 = 0)$$

$$SH^0 \cong \mathbb{K}[\theta_0, \theta_1, \theta_2, \theta_3]$$

$$(\theta_0 \cdot \theta_2 = 1, \theta_1 \theta_3 = 1)$$
If $D$ is a smooth divisor and $(\rho, \nabla)$ is a Hermitian structure on $ND$, then Weinstein’s tubular neighborhood theorem shows that there is an embedding from $\psi : U \subset ND \to M$ such that

$$\psi^*(\omega) = \pi^*(\omega|_D) + \frac{1}{2} d(\rho \alpha)$$

where $\alpha$ is the connection one-form.

In the normal crossings case, McLean-Tehrani-Zinger have introduced the notion of a ”regularization, ” which is essentially a system of compatible Weinstein tubular neighborhoods $\psi_i : U_i \to M$ which intersect nicely.

MTZ have shown that one can always deform $\omega$ (in the same cohomology class, keeping all $D_i$ symplectic) so that a regularization exists. McLean has further demonstrated that one can find a primitive $\theta$ for $\omega|_X$ which has some nice normal form.
We let $\tilde{X}$ be a small rounding of the corners of $M \setminus \bigcup_i (U_i, \varepsilon)$ where $\varepsilon > 0$ is some small real number and $U_i, \varepsilon$ is the region where $\frac{\rho_i}{\kappa_i / 2\pi} \leq \varepsilon^2$. This is a Liouville domain with Liouville coordinate $R$.

We want to take $H^\lambda$ to be a smoothing of:

Define

$$SH^*(X) := \lim_{\lambda \to 0} HF^*(X; H^\lambda)$$
Having made these choices, it is easy to calculate periodic orbits because the flow preserves the fibers of $U_i := \cap_{i \in I} U_i \rightarrow D_l$. The orbits come in connected families $\mathcal{F}_v$ which wind around the divisors with multiplicity $v$. For example, in the smooth case divisor one has

- constant orbits in the interior of $\bar{X}$.
- orbits which wind around the divisor $v > 1$ times.

**Calculation:** If $x_0 \in \mathcal{F}_v$, its action can be made arbitrarily close to

$$A_{H^\lambda}(x_0) \approx -w(v)(1 - \epsilon^2/2) \quad (5)$$

where $w(v) = \sum_i \kappa_i v_i$. Thus if $\epsilon$ is small, the filtration by $w(v)$ is essentially the same as the action filtration (up to sign).
Recall that a PSS solution asymptotic to an orbit $x_0$ is a map $u : \mathbb{CP}^1 \setminus \{0\} \to M$ satisfying a variant of Floer’s equation:

$$(du - X_{H^\lambda} \otimes \beta)^{0,1} = 0$$

(6)

(where $(0, 1)$ is taken with respect to some $J_S$) such that

$$\lim_{s \to -\infty} u(\varepsilon(s, t)) = x_0$$

(7)

In the last equation we are using the cylindrical coordinates (defined away from $z = \infty$) $\mathbb{R} \times S^1 \to S$

$$\varepsilon : (s, t) \to e^{2\pi(s+it)}.$$

$$\left(du - X_{H^\lambda} \otimes \beta\right)^{0,1} = 0$$

$$\mathcal{B} = p(s) dt$$

$$\frac{\partial (du)^{0,1}}{\partial t} = 0$$
Log PSS moduli spaces

**Definition**

Suppose $x_0$ is an orbit of $H^\lambda$ in $X$. Then a log PSS solution of multiplicity $v$ is a solution such that

- $u$ does not intersect $D$ anywhere except for at $z = \infty$.
- The intersection multiplicity of $u$ with $D_i$ at $z = \infty$ is $v_i$.

The virtual dimension

$$\text{vdim}(\mathcal{M}(v, x_0)) = \deg(x_0)$$

The (topological) energy is

$$E_{\text{top}}(u) = w(v) + A_{H^\lambda}(x_0)$$
Possible “bad” degenerations

There are two kinds of possible degenerations that we want to avoid

- Breaking along orbits in D.
- Sphere bubbling (both at $z = \infty$ and at other points in the domain)
It turns out that the breaking along orbits in D can be excluded provided one takes $\lambda > w(v)$.

If one considers only "low energy" moduli spaces i.e. those where $w(x_0) = w(v)$, then sphere bubbling is excluded by energy constraints.

Using the winding filtration, we have multiplicative spectral sequence

$$E_{kp}(\xi) \simeq \frac{1}{2} \omega(v) \xi^2$$

By counting only low energy moduli spaces, Ganatra and I defined a map

$$\text{PSS}_{\text{log}}^{\text{low}} : \mathcal{S}\mathcal{R}(\Delta(D)) \cong E^{p,-p}_{1}$$

which we proved to be an isomorphism.
We want to define an additive isomorphism:

$$\text{PSS}_{\text{log}} : \mathcal{A} \cong \mathcal{SH}^0(X)$$  \hspace{1cm} (8)

by count all moduli spaces with \(\deg(x_0) = 0\), not just the low energy ones.

- The sphere bubbles are hard to control if one thinks naively about the usual Deligne-Mumford compactification.
- The main idea is to construct a compatification of stable log PSS solutions following ideas/analysis of M. Tehrani. This has the property that all of the strata with sphere bubbles lie in virtual codimension 2 (hence in our case have negative virtual dimension).
- To “regularize” the boundary strata we adapt Cieliebak-Mohnke’s approach of stabilizing divisors.
Consider the case of a smooth divisor $D$ and suppose for simplicity we have some cylindrical Lagrangians $L_0, L_1$ such that

- $\pi(L_0), \pi(L_1)$ are embedded
- $\pi(L_0) \cap \pi(L_1)$ transversely

For each intersection point $y \in \pi(L_0) \cap \pi(L_1)$, we have chords $x_{y,v}$ where $x_{y,0}$ is a "short" chord and $x_{y,v}$ is given by taking that chord and spinning it $v$ times around $D$. We can arrange our data so that to, to lowest order, we have

$$\theta_v \cdot x_{y,0} = x_{y,v} + \cdots$$

(9)

where $\cdots$ denotes higher action terms.
Maximally degenerate setting

A pair \((M, D)\) is called maximally degenerate if \(D\) has a zero dimensional stratum. In this setting, one consequence of our result is:

**Proposition**

Let \((M, D)\) be a maximally degenerate Calabi-Yau pair of dimension \(n\). Suppose that \(\text{char}(k) = 0\), \(\text{Spec}(SH^0(X))\) is a reduced \(n\)-dimensional scheme of finite type which has Gorenstein singularities. Furthermore, it is Calabi-Yau.

- Main idea: use the fact that \(SH^0(X)\) is a deformation of \(SR(\Delta(D))\).
- Proving that \(SR(\Delta(D))\) is Gorenstein uses a deep result of Kollar-Xu on the topology of the dual intersection complex of a Calabi-Yau pair.
The above result makes it plausible that $\text{Spec}(SH^0(X))$ is closely related to a mirror variety of $X$. To be precise, suppose a mirror $Y$ to $X$ existed, then by classical algebraic geometry (baby verison of Zariski’s main theorem), one can show that $Y$ is a crepant resolution of singularities of $\text{Spec}(SH^0(X))$ (in particular these varieties are birational). However, there are two problems with this:

- Crepant resolutions are not unique in dimension $\geq 3$. So one needs more data to single out a crepant resolution.
- A crepant resolution may not even exist.
Definition

Given an affine log Calabi-Yau variety $X$, a homological section is an embedding $\pi^* : \text{Perf}(SH^0(X)) \hookrightarrow \mathcal{W}(X)$.

Remark

There are fairly explicit geometric criteria for when a Lagrangian $L_0$ determines a homological section.

Lemma

Suppose $X$ is equipped with a homological section. Then $(\mathcal{W}(X), \pi^*)$ is a categorical crepant resolution of $\text{Spec}(SH^0(X))$. 
A little more on log compactification

Assume $J_0$ is "split" near $z=\infty$

modelled on rooted trees $T$ in one log
- If $u_i((Dp')) \subset D_i$

then that component is decorated with a meromorphic section

$$[[i]] \subset \Gamma_{\text{mero}} (u^*(ND_i))^{\times}$$

with zeroes/poles only at nodes.

- For each marked point $\xi_{n,i}$ get an order function.
- Require

\[ \text{ord}(z_{n,n'}) = -\text{ord}(z_{n',n}) \]

this is the "pre-log" space

\[ \text{obj: } M^{\text{pre-log}}(v, x_0, T) \to \mathbb{G}(T) \]

\[ M^{\text{log}}(v, x_0, T) \]

\[ \text{a)} \text{ obj}_T(u) = 1 \]

\[ \text{b)} \text{ "a tropical condition"} \]
\[ \Theta_2 \cdot \Theta_3 = (1 + \Theta_0)^r \]
[BSV]

\( QH^* \) is a deformation of \( SH^*(X) \)

The deformation is described by

\[
\sum_{i} PSS_{s_{i}}(\Theta_{i}) = B
\]