Twisted generating functions and the nearby Lagrangian conjecture

(zoominar 26/02/21)

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0) Introduction

<u>Nearby Lagrangian conjecture</u>: Let M be a closed manifold and $L \subset T^*M$ is a closed exact Lagrangian submanifold. Then L is hamiltonian isotopic to the zero-section.

- Know to hold for $M = S^1$, S^2 and T^2.
- $L \to M$ is a homotopy equivalence (Abouzaid - Kragh via Floer theory, Guillermou via microlocal sheaf theory)
- If the conjecture holds, then L admits a generating function quadratic at infinity

(Sikorav's theorem)

- L admits a "generating sheaf" (Guillermou)

<u>Open question</u>: Does L admit a generating function (g.f.)?

<u>Thm</u> (Abouzaid, C, Guillermou, Kragh):
L admits a <u>twisted</u> g.f. of <u>tube type</u>.

<u>Cor</u>: The stable Lagrangian Gauss map
$L \longrightarrow U/O$ vanishes on all homotopy groups.

<u>Cor</u>: If $M = S^n$ or a homotopy sphere,
we obtain $L \to U/O$ is nullhomotopic and construct a genuine g.f.

- Work in progress with D. Alvarez-Gavela:
new restriction on the smooth structure of L. In particular, which $\Sigma^n \hookrightarrow T^*S^n$?

1) (Twisted) generating functions

$\mathcal{U} \underset{\text{open}}{\subset} M \times \mathbb{R}^n \qquad f : \mathcal{U} \longrightarrow \mathbb{R}$

$$\Sigma_f = \left\{ (q,v) \in M \times \mathbb{R}^n \, , \, \frac{\partial f}{\partial v}(q,v) = 0 \right\}$$

regular equation

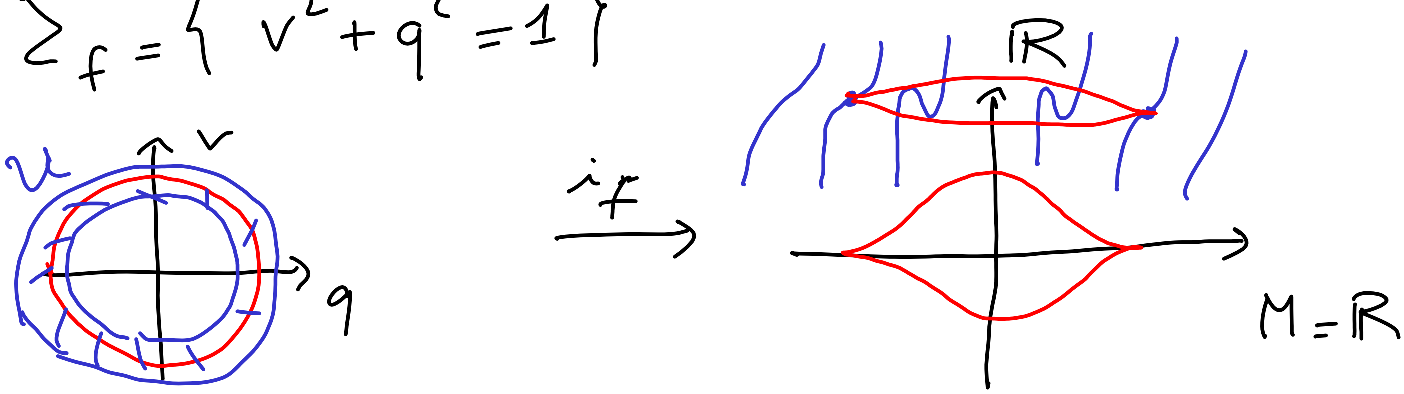
<u>Fact</u>: $\Sigma_f \xrightarrow{i_f} J^1 M = T^*M \times \mathbb{R}$

$(q,v) \longmapsto \left(q, \frac{\partial f}{\partial q}(q,v), f(q,v)\right)$

is a Legendrian immersion, said to be "generated by f".

<u>Ex</u>: $f : \mathbb{R}^2 \to \mathbb{R} \qquad f(q,v) = v^3 + 3(q^2 - 1)v$

$\Sigma_f = \{ v^2 + q^2 = 1 \}$

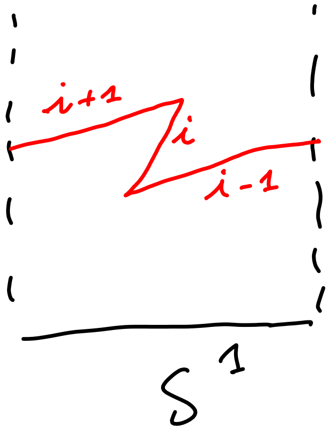


Think of f as a family $(f_q)_{q \in M} : \mathbb{R}^n \to \mathbb{R}$.

Cerf diagram of (f_q) = front projection of $i_f(\Sigma_f)$

For the moment $\mathcal{U}$ can be a tiny neighborhood of Σ_f.

- Not all Legendrians admit a g.f.



$i-1 \neq i+1$

First Maslov class obstruction

$$\mu_1 \in H^1(L, \underset{\pi_1 U/O}{\overset{\shortparallel}{\mathbb{Z}}})$$

μ_1 is the obstruction to the triviality of the Gauss map $L \to U/O$ on the 1-skeleton.

<u>Thm</u> (Giroux, Latour 90): A Legendrian immersion $L \to J^1M$ admits a g.f. $\iff$ $L \to U/O$ is nullhomotopic.

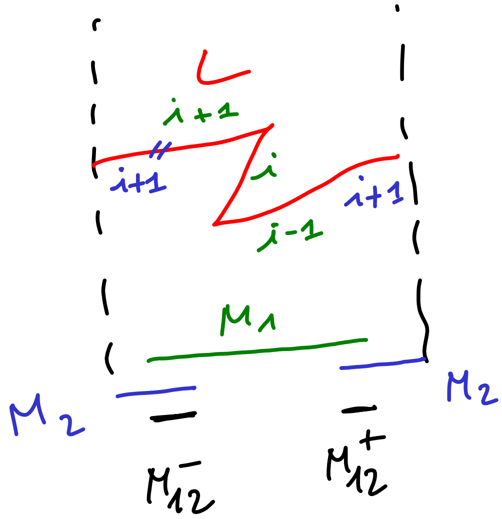
- For L nearby Lagrangian, $L \overset{?}{\simeq} U/O$ still open

but we know

$$\begin{tikzcd} L \arrow[r] \arrow[dr, "\sim"'] & U/O \\ & M \arrow[u, dashed, "\exists"'] \end{tikzcd}$$

we will use the map $M \to U/O$ to "twist" a g.f.

Ex:



$M = S^1 = M_1 \cup M_2$

A twisted g.f. is:

$f_1 : M_1 \times \mathbb{R}^{n_1} \longrightarrow \mathbb{R}$ g.f

$f_2 : M_2 \times \mathbb{R}^{n_2} \longrightarrow \mathbb{R}$

$n_2 = n_1 + 2$

$q_{12} : M_{12} \times \mathbb{R}^2 \longrightarrow \mathbb{R}$

fiberwise non-degenerate quadratic forms

$$q_{12}(q, u, v) = \begin{cases} u^2 + v^2 & , x \in M_{12}^- \\ -u^2 - v^2 & , x \in M_{12}^+ \end{cases}$$

so that $f_1 \oplus q_{12} = f_2$

Def: A twisted g.f. is:

- $(M_i)_{i \in I}$ open cover of M, I ordered.
- $\forall i$, $f_i : M_i \times \mathbb{R}^{n_i} \longrightarrow \mathbb{R}$ g.f.
- $\forall i < j$, $q_{ij} : M_{ij} \times \mathbb{R}^{n_{ij}} \longrightarrow \mathbb{R}$ fiberwise non-deg quadratic form

satisfying:

- $\forall i < j$, $f_i \oplus q_{ij} = f_j$ on M_{ij}.
- $\forall i < j < k$, $q_{ij} \oplus q_{jk} = q_{ik}$ on M_{ijk}.

The twisting datum (n_i, q_{ij}). It looks like a bundle except q_{ij} is valued in a monoid:

$$Q = \bigsqcup_{n \in \mathbb{N}} \{\text{non-deg quadratic forms } \mathbb{R}^n \to \mathbb{R}\}$$

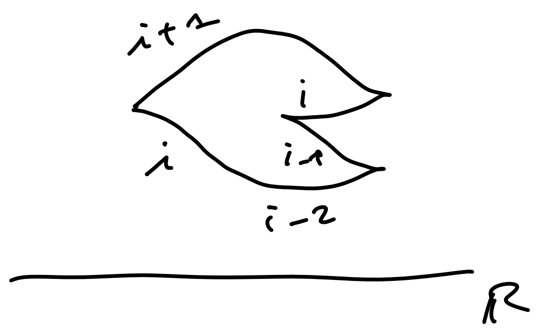
with operation = direct sum.

There is a classifying space $B(\mathbb{Z}, Q)$ and it turns out (Bott periodicity):

$$B(\mathbb{Z}, Q) \simeq U/O$$

So the twisting datum is precisely given by a map $M \to U/O$, so this encodes perfectly the Gauss map $L \to U/O$ for a nearby Lagrangian since $L \xrightarrow{\sim} M$.

<u>NB</u>: If $L \xrightarrow{O} M$, twisting cannot help.

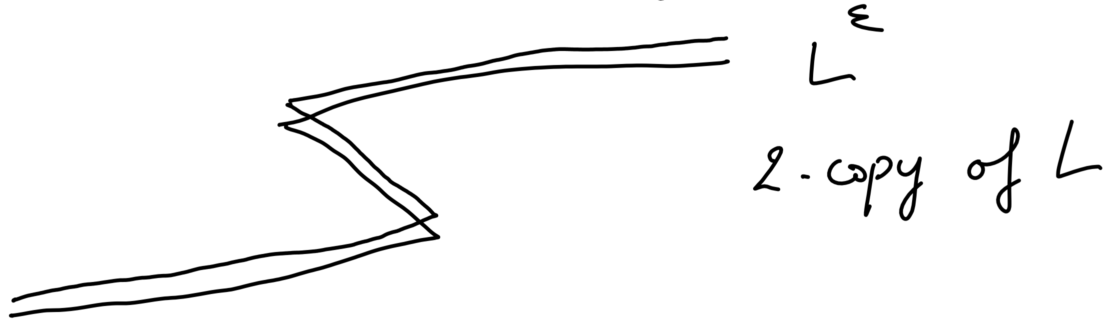


no g.f. exist even twisted.

<u>Thm</u>: $L \to J^1 M$ Legendrian immersion admits a twisted g.f $\Leftrightarrow$ $L \to U/O$ factors through M.

Problem: This notion of g.f is too weak.

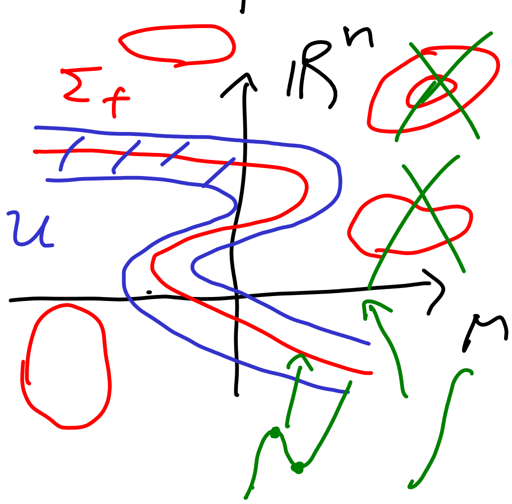
② Guillermou's doubling trick.



- Any "microlocal sheaf" on L can be converted into a genuine sheaf for L^ε.
- The same principle holds for g.f. where "microlocal g.f." means a germ of g.f. near the singular set.

How does that work?

Let $f : U \to \mathbb{R}$ be generating L.



- Extend f arbitrarily to $\tilde{f} : M \times \mathbb{R}^n \to \mathbb{R}$

get extra undesired Legendrian

- Set $f^\varepsilon : M \times \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$

$$f^\varepsilon(q,v,w) = f(q,v) + w^3 - \varepsilon \alpha(q,v)\, w$$

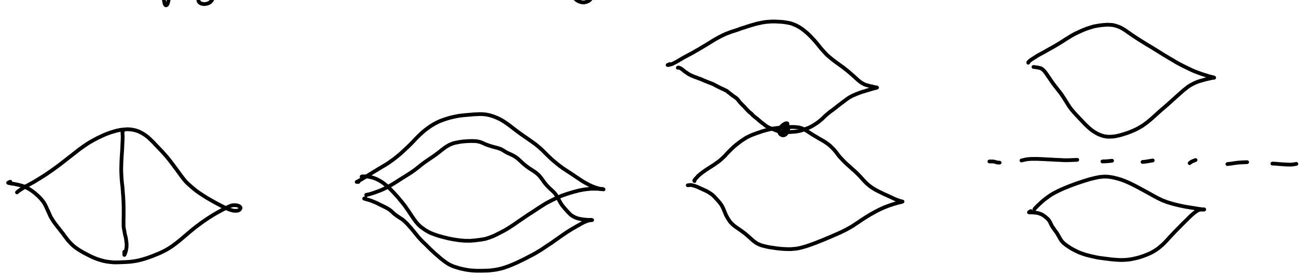
$$\alpha(q,v) = \begin{cases} 1 & \text{near } \Sigma_f \\ <0 & \text{away from } \mathcal{U} \end{cases}$$

so $\dfrac{\partial f^\varepsilon}{\partial w} > 0$ if $(q,v) \in \mathcal{U}$.

and f^ε generates L^ε.

Moreover f^ε is well-behaved (essentially $f^\varepsilon = w$ at infinity): linear at infinity.

<u>Next step</u>: Let ε increase: L^ε is a regular homotopy with crossings $\overset{1:1}{\longleftrightarrow}$ Reeb chords of L.



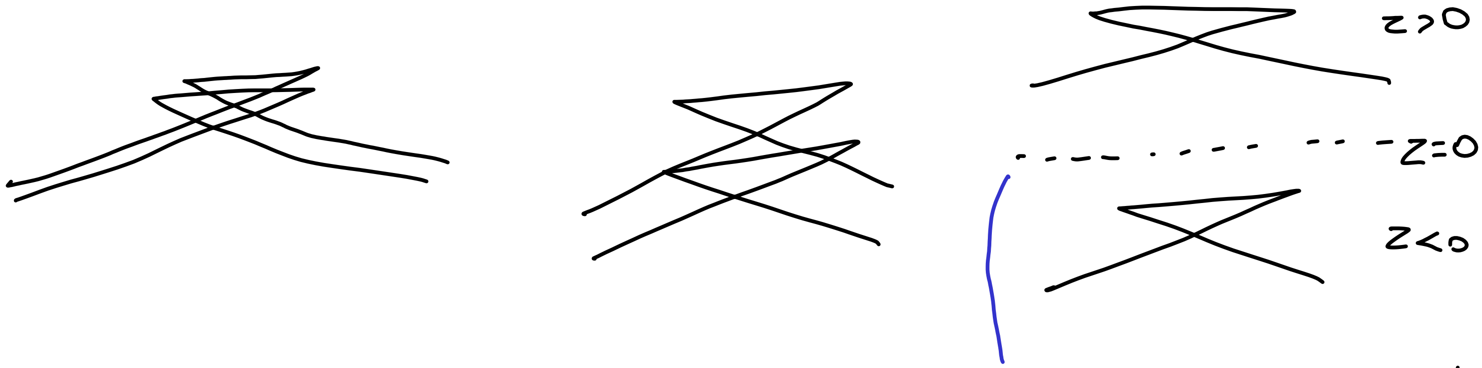
Separated 2-copy

- If L is a nearby Lagrangian there is no Reeb chord so L^ε is a Legendrian isotopy.

<u>Homotopy lifting property</u> (Chekanov's theorem):

g.f. persist (up to stabilization) under Legendrian isotopy.

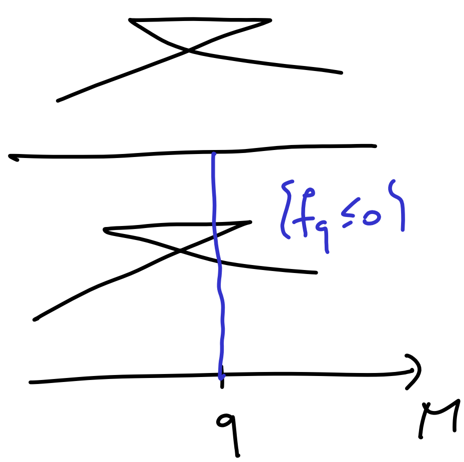
- also true for twisted g.f.



→ get a twisted g.f. for L by restricting to $\{z<0\}$.

How does it look like?

③ Tube spaces and Bölstedt's theorem



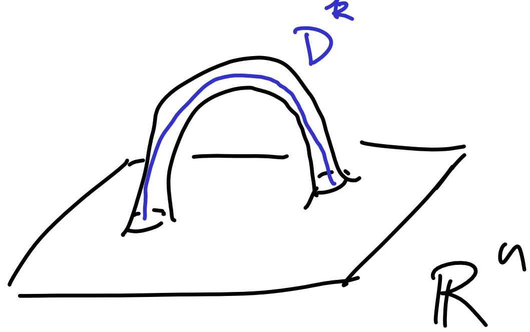
- We show: $\forall q \in M$

$$H_*(f_q \leq 0,\ f_q \leq -\infty) = \mathbb{Z}$$

- $\{f_q = -\infty\} \simeq \{w = -\infty\} \simeq \mathbb{R}^n$

By Morse-Smale theory:

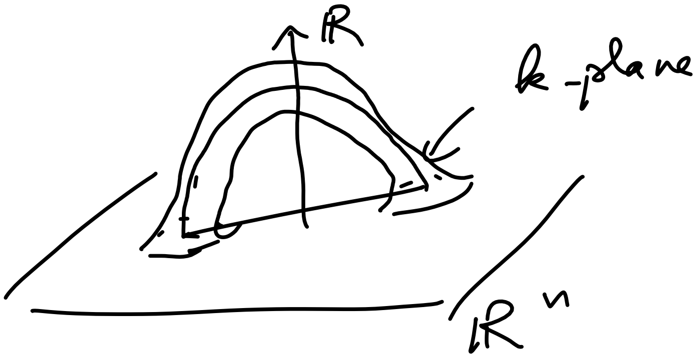
$\{f_q \leq 0\} = \mathbb{R}^n \times (-\infty, 0] \cup 1$ trivial handle of index k



Waldhausen considered the space of all such hypersurfaces $\{f_q = 0\}$ in $\mathbb{R}^{n+1}$, $T_{k,n}$.

It comes with a map:

$$G_{k,n} \longrightarrow T_{k,n}.$$ rigid tube map



Stably $k, n \to \infty$

$$BO \longrightarrow T_\infty.$$

<u>Bökstedt</u>: $\pi_i BO \hookrightarrow \pi_i T_\infty$ injective

Going back to our twisted g.f. (f_i, q_{ij})

$$M_i \xrightarrow{\{f_i = 0\}} T_\infty \qquad f_i \oplus q_{ij} = f_j$$

$\rightsquigarrow$ assemble into a map $M \to B(T_\infty, Q)$
$$B(T_\infty, Q) = T_\infty / BO$$

which factors the Gauss map

$$\begin{tikzcd}
0 \arrow[r] & \pi_i BO \arrow[r, hook] & \pi_i T_\infty \arrow[r] & \pi_i T_\infty/BO \arrow[r, "0"] & \pi_i U/O \\
 & & & \pi_i M \arrow[u, dotted, "\text{twisted g.f.}"] \arrow[ur, "\text{Gauss}"'] &
\end{tikzcd}$$

So $L \simeq M \to U/O$ vanishes on all π_i.