The large-scale geometry of Hofer's metric

joint work with Dan Cristofaro-Gardiner, Vincent Humilière

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Symplectic Zoominar
Mar. 5, 2021

Introduction: Hofer's metric

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- $d_H(\varphi,\psi) \leq d_H(\varphi,\theta) + d_H(\theta,\varphi)$.
- non-degeneracy: $d_H(\varphi,\psi) = 0 \iff \varphi = \psi$. (Hofer, Polterovich, Lalonde-McDuff)

The large scale geometry of Hofer's metric
&
two old questions.

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Eg: 1. $\mathbb{Z} \overset{QI}{\sim} \mathbb{R}$, 2. $\mathbb{R} \overset{QI}{\nsim} \mathbb{R}^2$, 3. X bdd $\implies X \overset{QI}{\sim} pt$.

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Conclusion: $(\mathrm{Ham}(\mathbb{S}^2), d_H)$ is big.

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- Polterovich: QI embedding of $\mathbb{R}$. (1998)

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Question (Fathi, late 70s)

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  - 2020-2021: $\mathrm{Homeo}_c(D, \omega), \mathrm{Homeo}_0(\mathbb{S}^2, \omega)$ (our work)
    - No known natural homomorphism.
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The QI embeddings

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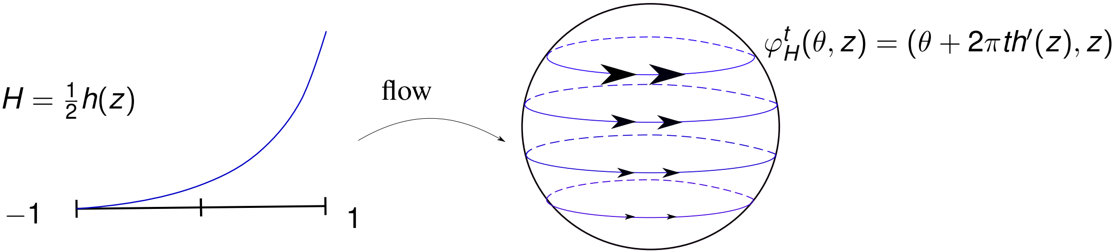

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Discs: $D_i = \{(z, \theta) : 1 - \frac{1}{i+1} \leq z \leq 1\}$. Note: $D_i \supset D_{i+1}$, Area$(D_i) = \frac{1}{2(i+1)}$.

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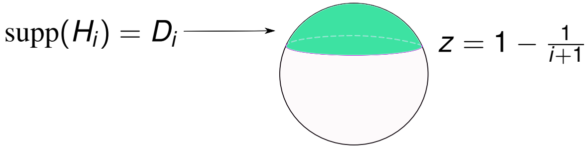

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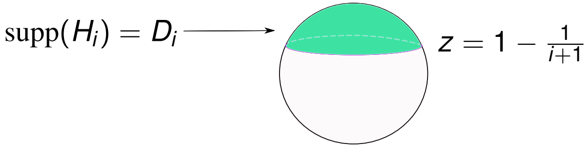

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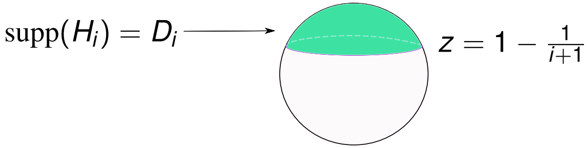

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Theorem: Φ is a QI embedding.

Outline of argument in the $n = 2$ case

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Corollary: $\mathrm{Ham}(\mathbb{S}^2) \overset{\mathrm{QI}}{\not\simeq} \mathbb{R}$.

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Monotone twist formula: H monotone twist Hamiltonian. Then,

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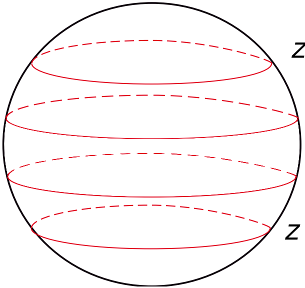

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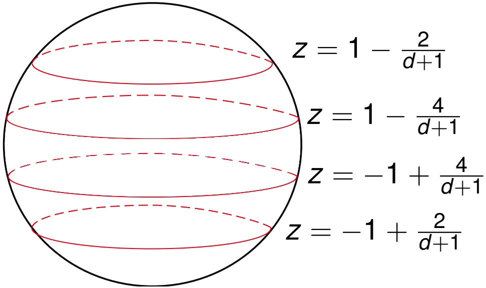

Linearity for monotone twists: $\mu_d(\varphi_{H_1}^{t_1} \circ \varphi_{H_2}^{t_2}) = t_1 \mu_d(\varphi_{H_1}^1) + t_2 \mu_d(\varphi_{H_2}^1)$.

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So, take $C_1 = \frac{1}{\|A^{-1}\|_{op}}$, hence the lower bound.

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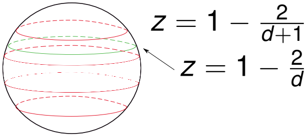

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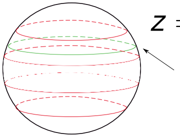

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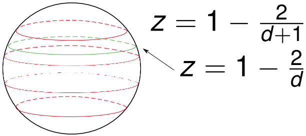

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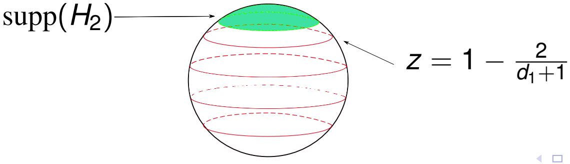

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- Put it all together, as explained above.

A word on μ_d

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FHomeo being proper means : $\exists$ homoes which are infinitely far from diffeos.

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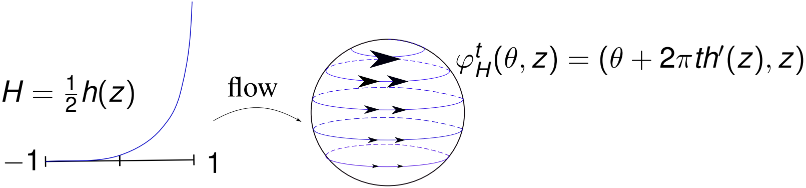
The infinite twist

$H : \mathbb{S}^2 \to \mathbb{R}$ of the form $H(\theta, z) = \frac{1}{2}h(z)$, where $h' \geq 0, h'' \geq 0$.



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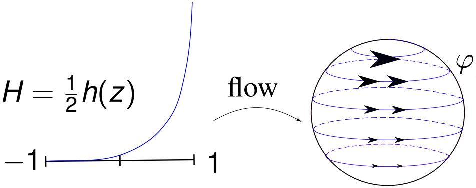

$\varphi_H^t(\theta, z) = (\theta + 2\pi t h'(z), z)$

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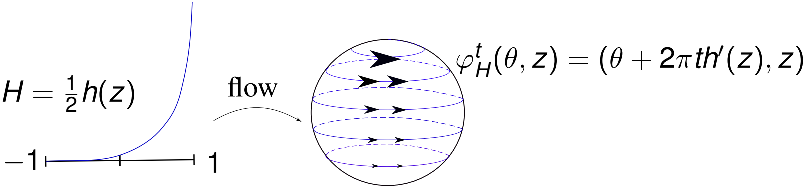

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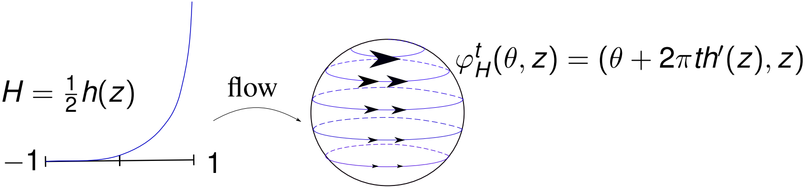

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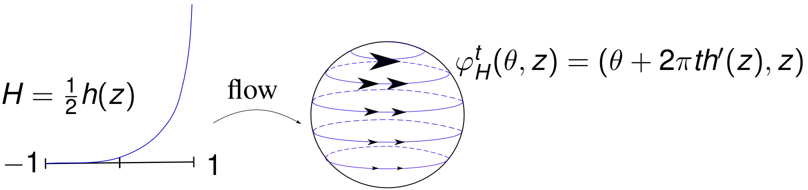

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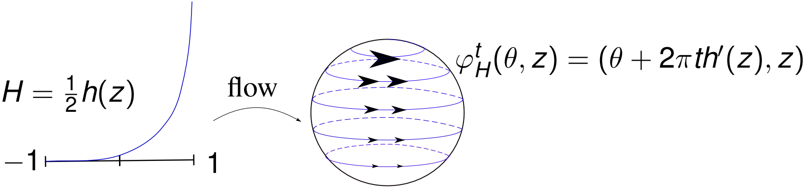

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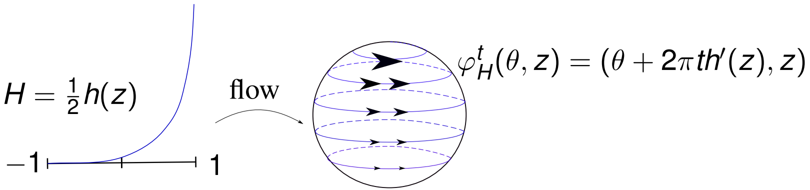

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Comparison with our previous paper

We used PFH spectral invariants to show $\mathrm{Homeo}_c(D^2, \omega)$ is not simple in previous work. A key new ingredient here is the construction of invariants η_d, μ_d, c_d on $\mathrm{Ham}(\mathbb{S}^2, \omega)$.

Challenge: need invariants that depend only on the time-1 map, not the choices involved in the construction.

Bonus:

Periodic Floer Homology:

impressionistic sketch of the construction.

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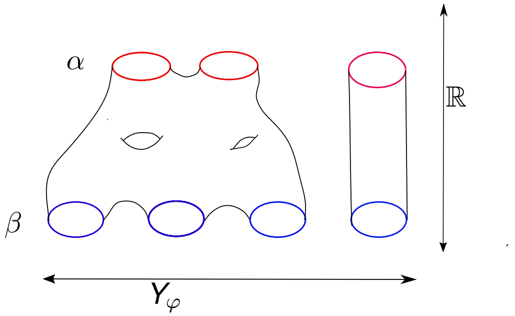
Lee-Taubes: $PFH(\varphi)$ independent of choices of J, φ.

A J-hol curve contributing to $\langle \partial\alpha, \beta \rangle$



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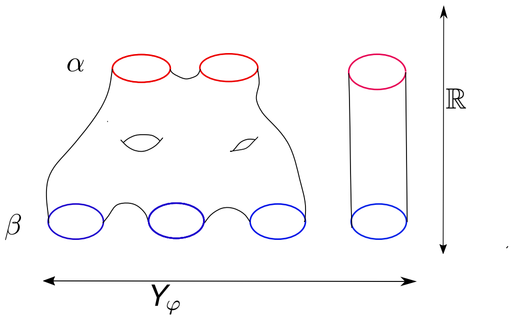

$\langle \partial\alpha, \beta \rangle := \#$ maps $u : (\Sigma, j) \to (\mathbb{R} \times Y_\varphi, J)$ such that

- J holomorphic: $du \circ j = J(u)du$.
- Asymptotic to α and β.
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Remark: d corresponds to the homology class of the orbit set.

Thank you!

Appendix: More details on PFH.

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- Lee-Taubes: Does not depend on the choices of J, φ.

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- J holomorphic: $du \circ j = J(u) du$.
- Asymptotic to α and β.
- "ECH index" $I = 1$

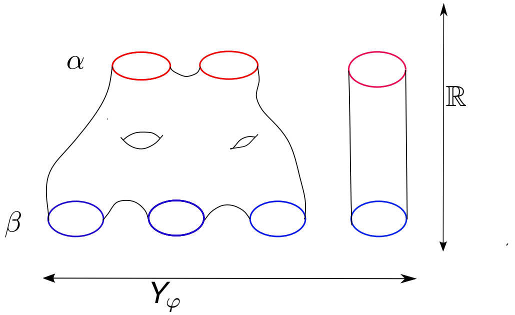

Figure: A J-hol curve contributing to $\langle \partial\alpha, \beta \rangle$.

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Degree of $\alpha = \{(\alpha_i, m_i)\}$: the homology class

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$\widetilde{PFH}(\varphi, d)$: “Twisted" version of PFH. Has an action filtration.

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Consider

$$i^a : \widetilde{PFH}^a(\varphi, d) \to \widetilde{PFH}(\varphi, d)$$

induced by inclusion.
Define

$$c_d(\varphi) := \inf\{a : \sigma \in \mathrm{Im}(i^a)\}.$$

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The distinguished class σ: the non-zero class in $\widetilde{PFH}_*(\varphi, d)$.