Lagrangian configurations and Hamiltonian maps

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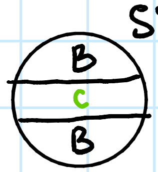
joint work with Leonid Polterovich

$\mathrm{Ham}(M,\omega)$ – group of Hamiltonian diffeomorphisms.
(M,ω) – closed.

$A \subset M$ (closed) is displaceable if $\varphi A \cap A = \emptyset$ for some $\varphi \in \mathrm{Ham}(M,\omega)$.

<u>Question</u> (Polterovich)

S^2 $\mathrm{Area}(S^2) = 1$



$\mathbb{L}^0_{2,B}$, $B > \frac{1}{4}$

$2B > \frac{1}{2}$

$C = 1 - 2B$

① Fact: $\mathbb{L}^0_{2,B}$ is non-displaceable

② Fact: $\mathbb{L}^0_{2,B}$ is stably displaceable:
$\mathbb{L}^0_{2,B} \times 0_{S^1} \subset S^2 \times T^*S^1$ is displaceable

$\mathbb{L}_{2,B}$ =

③ Is it true that $\mathbb{L}^0_{2,B} \times$ equator $\subset S^2 \times S^2(2a)$ is non-displaceable when a is small?

area $2a$

④ Qn: find the best a.

$= M_a$

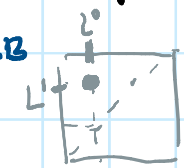
<u>Thm (Mak–Smith, 2019)</u> $\mathbb{L}_{2,B} \subset S^2 \times S^2(2a)$ is non-displaceable if $0 < a < 3B - 1$.

<u>Key idea</u>: ① $\mathbb{L}_{2,B} = L^0_{2,B} \sqcup L^1_{2,B}$ disjoint components

It becomes more rigid in $\mathrm{Sym}^2(M_a)$ symmetric square orbifold: $\{L^0_{2,B} \times L^1_{2,B}\} \subset \mathrm{Sym}^2(M_a) = M_a^2 / \mathbb{Z}/2\mathbb{Z}$.

square orbifold: $[L^0_{2,B} \times L'_{2,B}] \subset \mathrm{Sym}^2(M_a) = M_a^2/_{\mathbb{Z}/2\mathbb{Z}}$:

↳ $\mathcal{L}_{2,B}$



$(x,y) \mapsto (y,x)$

② This is because

there are more holomorphic disks with bdry on $\mathcal{L}_{2,B}$

$$(*) \quad u: D \to \mathrm{Sym}^2(M_a)$$
$$\partial D \to \mathcal{L}_{2,B}$$

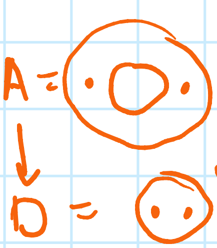
if Σ is a holomorphic branched cover of degree 2, $\pi_0 \partial\Sigma = 2$
$\downarrow \pi$
D

then a hol. map $v: \Sigma \to M_a$
$\partial\Sigma \to L_{2,B}$

yields such a hol. disk $(*)$:

$$z \in D \qquad u(z) = v(\pi^{-1}(z)) \in \mathrm{Sym}^2(M_a).$$

* this can be reversed: "tautological correspondence"

Key example: A =  annulus $A = \{\frac{1}{r} \leq |z| \leq r\}$

disk

$D \cong A/_{\mathbb{Z}/2}$

$z \mapsto \frac{1}{z}$

③ These disks allow new bulk deformations by "truly orbifold classes" in framework of Cho-Poddar, Fukaya-Oh-Ohta-Ono.

$\rightsquigarrow$ And hence $HF((\mathcal{L}_{2,B},\mathfrak{b},b),(\mathcal{L}_{2,B},\mathfrak{b},b)) \neq 0$
for some $\mathfrak{b}$-bulk, b-bounding cochain.
In particular $\mathbb{L}_{2,B} \subset M_a$ is not displaceable. ✓

<u>Viterbo, Oh, Leclercq-Zapolsky, FOOO:</u>

Lagrangian spectral invariants with Hamiltonian perturbation term: $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R})$ Hamiltonian

$L \subset M$ Lagrangian with $HF(L,L) \neq 0$

$\rightsquigarrow c(L, 1_L; H) \in \mathbb{R}$ satisfying good properties.

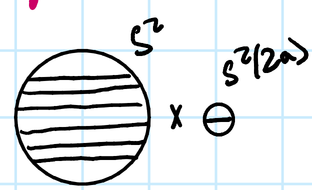
* calculates the smallest action level under which 1_L "appears" in $HF(L;H) \cong HF(L,L)$.

* $c(L, 1_L; H) \in \mathrm{Spec}(L;H)$ = action of a generator.
* If $H|_L \equiv c$ $\overset{\text{usually}}{\Rightarrow}$ $c(L, 1_L; H) = c$.

We investigate Hamiltonian metric rigidity and dynamics by means of spectral invariants coming from $\mathcal{L}_{2,B}$ and its generalizations $\mathcal{L}_{k,B} \subset \mathrm{Sym}^k(M_a)$.

$$\mathcal{L}_{k,B} = [L^0_{k,B} \times \dots \times L^{k-1}_{k,B}] \subset \mathrm{Sym}^k(M_a)$$

↑ to be specified later



$$H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M_a, \mathbb{R}) \rightsquigarrow c_{k,B}(H) = \frac{1}{k}\, c(\mathcal{L}_{k,B}, 1_{\mathcal{L}_{k,B}}; H^{(k)})$$

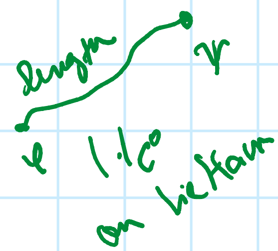
Hamiltonian on $\mathrm{Sym}^k(M_a)$ induced by $H(x_1) + \dots + H(x_k)$ on M_a^k

Gambaudo-Ghys et al: studied $Ham(\Sigma)$ by action on configuration space $Conf_k(\Sigma)$. We realize a Floer-theoretical version of this. Results on Hofer metric instead of hydrodynamic metrics.

Hofer, Viterbo, Polterovich, Lalonde-McDuff:

$Ham(M,\omega)$ carries a remarkable bi-invariant metric:

$$\underline{d_{Hofer}(\varphi,\psi)} = \inf_{\varphi_H^1 = \psi\varphi^{-1}} \underline{\int_0^1 \max_M |H(t,-)|\, dt}$$



φ_H^t generated by X_H^t, $\underline{\int H_t \omega^n = 0}$ $\forall t$.

$$H \in C^\infty(\mathbb{R}/\mathbb{Z} \times M, \mathbb{R})$$

Non-degenerate: $d(\varphi,\psi) > 0$ for $\varphi \neq \psi$.

Question: is $diam(Ham, d_{Hofer}) = \infty$? Open in general.

d_{Hofer} lifts to a pseudo-metric $\tilde{d}_{Hofer}$ on $\widetilde{Ham}(M,\omega)$.

universal cover.

Ostrover: $diam\ \tilde{d}_{Hofer} = +\infty$ for all (M,ω)-closed.

Question: is $\tilde{d}_{Hofer}$ non-degenerate? Open in general ☺

We obtain results in 2D and in 4D.

$\boxed{2D}$ Surfaces: $\Sigma_{g \geq 1}$ $diam(d_{Hofer}) = +\infty$ Lalonde-McDuff '95

S^2 $diam(d_{Hofer}) = +\infty$ Polterovich '98

Question: Which groups embed?

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Rigid embeddings: $(G,d) \xrightarrow[\text{morphism}]{\Phi} (\mathrm{Ham}, d_{Hofer})$

① isometric: $d_{Hofer}(\Phi(x),\Phi(y)) = d(x,y)$

② quasi-isometric:

$$\frac{1}{C} d(x,y) - K \le d_{Hofer}(\Phi(x),\Phi(y)) \le C\, d(x,y) + K$$

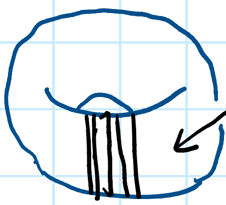
for all x,y $\quad C \geq 1,\ K \geq 0.$

① $\Sigma_{g\geq 1}$: i. $(\mathbb{R}^k, \|\cdot\|) \overset{q.i.}{\hookrightarrow} (\mathrm{Ham}(\Sigma), d_{Hofer})$ $\forall k$ Py '08

$\Sigma =$

ii. $(\mathbb{R}^\infty, \ell_\infty) \overset{q.i.}{\hookrightarrow} (\mathrm{Ham}(\Sigma), d_{Hofer})$ Usher '13

iii. $(C_c^\infty((a,b)), d_{C^0}) \overset{isom.}{\hookrightarrow} (\mathrm{Ham}(\Sigma), d_{Hofer})$ Polterovich '98, Zapolsky '13

Idea: 

foliate a neighborhood of a non-separating curve by similar curves and use symplectic rigidity.

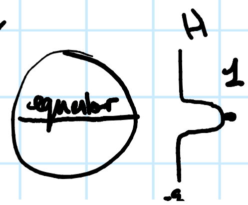
- different flavors for functions constant along each curve

i.,ii.,iii. extend to other classes of manifolds.

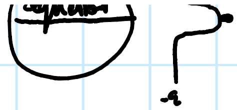
② S^2: $\mathbb{R} \hookrightarrow \mathrm{Ham}(S^2)$ quasi-isom. (Polterovich)

isom (Entov-Polterovich, Leclercq-Zapolsky)

S^2



$d_{Hofer}(\varphi_H^t, id) = |t|$



$d_{Hofer}(\varphi^t_H, id) = |t|$

Question (Kapovich-Polterovich, 2006): Is $Ham(S^2)$ in an R-tube around this $\mathbb{R} \subset Ham(S^2)$?

(for some $R>0$)

Thm^A (Polterovich-S., 2021):

$$\left(C^\infty_c((0,1)), d_{C^0}\right) \xrightarrow{\text{isom.}} \left(Ham(S^2), d_{Hofer}\right)$$

infinite-dimensional!

.... so no.

Cor (PS, 2021; Cristofaro-Gardiner-Humilière-Seyfaddini, 2021)

Also no, because

different techniques PFH

$$\forall k \qquad (\mathbb{R}^k, \|\cdot\|) \xrightarrow{\text{q.i.}} (Ham(S^2), d_{Hofer})$$

Thm^B (PS-2021) For all $U \subset S^2$ open proper

small-scale rigidity in Hofer's geometry

$$C^\infty_c((0,1)) \xrightarrow{\text{isom.}} \operatorname{Ker} Cal_U \subset Ham(S^2)$$

n=1

$$\varphi \in Ham_c(U), \quad Cal_U(\varphi) = \int_0^1 \int_U F_t \, \omega \, dt$$

(CGHS-2021) for $\mathbb{R}^k$ $\forall k$

$$F_t = 0 \text{ near } \partial U, \quad \varphi^1_F = \varphi.$$

Another 2D result:

Thm (CGHS, 2020) $G = \mathrm{Homeo}_{0,c}(\mathbb{D}^2, \text{area})$ not simple

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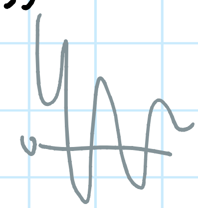
Proper normal subgroup: $\{\varphi \text{ s.t. } \varphi_i \xrightarrow{C^0} \varphi,\ d_{\text{Hofer}}(\varphi_i, id) \leq C\}$

$\mathrm{Homeo} \subset G^F = \mathrm{FHomeo}$

↑ Oh, Müller

CGHS: $\mathbb{R} \hookrightarrow G/G^F$

Thm^C (P.S., 2021): $C^1((0,\tfrac{1}{2}]) / C^1_{bdd}((0,\tfrac{1}{2}]) \hookrightarrow G/G^F$



So G is very non-simple.

$\boxed{4D}$

Question: $\mathrm{Ham}(S^2) \to \mathrm{Ham}(S^2 \times S^2(2a))$ Isometric embedding?

$\varphi \longmapsto \varphi \times id$ Quasi-isometric?

Lipschitz map

Thm^D (PS-2021) $\quad (\mathrm{Ham}(S^2), d_{Hofer})$, isom., $\varphi \mapsto \varphi \times id$

stabilization: $(C^\infty_c((a,b)), d_{C^0}) \xhookrightarrow{q.i.} (\mathrm{Ham}(S^2 \times S^2(2a)), d_{Hofer})$

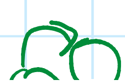
for a sufficiently small

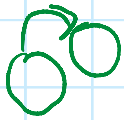
isom. on univ covers.

Lagrangian Packing

Seemingly unrelated question:

Let $L \subset M$ Lagrangian, displaceable:





$\varphi L \cap L = \emptyset$ for some $\varphi \in Ham$.

Question 1: how many Hamiltonian copies:
Packing

$\varphi_1 L, \varphi_2 L, \ldots, \varphi_k L$ of L can be pairwise disjoint: $\varphi_i L \cap \varphi_j L = \emptyset$, $i \neq j$?

Question 2: Fix $\varphi \in Ham$. How often is
Recurrence

$$\varphi^i L \cap L \neq \emptyset ?$$

Rmk: in dim 2, $\partial D = L$, D - disk of minimal area.

$\Rightarrow$ measure theory + combinatorics:

$\varphi^i L \cap L \neq \emptyset$ with frequency $\geq \frac{\text{area}(D)}{\text{area}(M)}$.

Thm (Ginzburg-Gürel, 2018) $\varphi \in Ham(\mathbb{C}P^n)$
Hamiltonian pseudo-rotation $|Per(\varphi)| = n+1$

$$\Rightarrow \liminf_{N \to \infty} |\{1 \leq i \leq N \mid \varphi^i L \cap L \neq \emptyset\}| / N \geq C_\varphi \cdot \varepsilon_L^n.$$

$\forall L \subset \mathbb{C}P^n$ closed $\quad C_\varphi > 0$, $\varepsilon_L > 0$

symplectic invariant

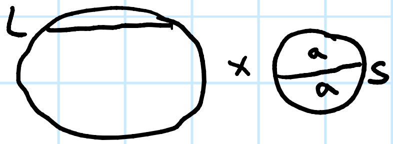
First cases of examples of L that works for all φ:

Thm^E (PS-2021) Let $L \subset S^2$ be ∂D, $\frac{1}{k+1} < \text{area}(D) < \frac{1}{k}$

$\underline{\text{Thm}^E}$ (PS-2021) Let $L \subset S^2$ be ∂D, $\frac{1}{k+1} < \text{area}(D) < \frac{1}{k}$

$S \subset S^2(2a)$ equator. $B = \ldots$ $k \geq 2$

If $a < \frac{(k+1)B - 1}{k-1}$ "small" then



(1) can pack $M_a = S^2 \times S^2(2a)$ by k copies of

$\Lambda = L \times S$ easy: already in S^2

(2) cannot pack M_a by $(k+1)$ copies of Λ.

$\underline{\text{Thm}^F}$ (PS-2021) For same $\Lambda \subset M_a$,

$$\liminf_{N \to \infty} |\{i < N \mid \varphi^i \Lambda \cap \Lambda \neq \phi\}| / N \geq \frac{1}{k}.$$

For all $\varphi \in \text{Ham}(M_a)$.

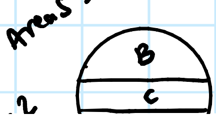
$\text{Thm}^E \Rightarrow \text{Thm}^F$: combinatorics.

Method of proof: Lagrangian estimators

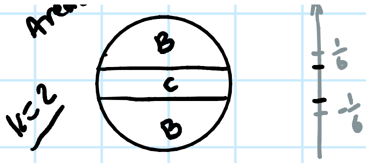
$\approx$ certain Lagrangian spectral invariants supported on a collection of disjoint Lagrangians.

$z : S^2 \to [-\frac{1}{2}, \frac{1}{2}]$ moment map for standard S^1-action

Area $S^2 = 1$



$2B + (k-1)C = 1$

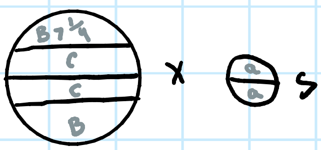


$$2B+(k-1)C = 1$$
$$B > C$$

$$\mathbb{L}^0_{k,B} = \bigsqcup_{j=0}^{k-1} \underbrace{z^{-1}\left(-\tfrac{1}{2}+B+jC\right)}_{L^{0,j}_{k,B}} \subset S^2$$

$S \subset S^2(2a)$ equator

$$\mathbb{L}_{k,B} = \mathbb{L}^0_{k,B} \times S = \bigsqcup_{j=0}^{k-1} L^j_{k,B} \subset M_a$$
$$L^j_{k,B} = L^{0,j}_{k,B} \times S$$



Recall: $M_a = S^2 \times S^2(2a)$

<u>Theorem</u> (PS-2021): Let $0 < a < B-C$, $k \geq 2$. $a, B, C \in \mathbb{Q}$

There exist maps $\mu_{k,B} : \widetilde{\mathrm{Ham}}(M_a) \longrightarrow \mathbb{R}$

satisfying a list of properties including: $\forall \varphi, \psi \in \widetilde{\mathrm{Ham}}(M_a)$

1. <u>Hofer</u> - <u>Lipschitz</u>: $|\mu_{k,B}(\varphi) - \mu_{k,B}(\psi)| \leq \tilde{d}_{Hofer}(\varphi,\psi)$

2. <u>Lagrangian control</u>: If $H_t|_{L^j_{k,B}} \equiv c_j(t)$ $\in \mathbb{R}$ $0 \leq j < k$
$$\Rightarrow \mu_{k,B}(\varphi_H) = \frac{1}{k}\sum_{0 \leq j < k} \int_0^1 c_j(t)\,dt$$
$\varphi_H = [\{\varphi_H^t\}]$

3. <u>Conjugation invariance</u>: $\mu_{k,B}(\psi\varphi\psi^{-1}) = \mu_{k,B}(\varphi)$

4. <u>Positive homogeneity</u>: $\mu_{k,B}(\varphi^m) = m\,\mu_{k,B}(\varphi)$ $m \in \mathbb{Z}_{>0}$

5. <u>Calabi property</u>: H supported in $\mathbb{R}/\mathbb{Z} \times U$, $U \cap \mathbb{L}_{k,B} = \emptyset$
$$\Rightarrow \mu_{k,B}(\varphi_H) = -\frac{1}{\mathrm{Vol}(M_a)}\,\mathrm{Cal}(\varphi_{H,U})$$
$\widetilde{\mathrm{Ham}}_c(U)$

6. $H = F \oplus 0 \Rightarrow \mu_{k,B}(\varphi_H)$ $\mathrm{Ham}(S^2)$

⑥ $H = F \oplus 0 \Rightarrow \mu_{k,B}(\varphi_H)$ depends only on $\varphi_F^1 \in \mathrm{Ham}(S^2)$ $\qquad \widetilde{\mathrm{Ham}}_c(U)$

{ We use the freedom to choose B and k to prove our results.

Idea: $\mu_{k,B}(H) = \lim_{m \to \infty} \frac{1}{m} c_{k,B}(H^{\# m})$

$(C_c^\infty((0,1)), d_{C^0}) \xrightarrow{\text{isom.}} (\mathrm{Ham}(S^2), d_{\mathrm{Hofer}})$

Thm A: take $F = h \circ z$, for an even function $h \in C_c^\infty((-\frac{1}{6}, \frac{1}{6}))$.

Extend it to have 0 mean without changing $|h|_{C^0}$.

Enough to prove $d_{\mathrm{Hofer}}(\varphi_F^1, id) = |h|_{C^0}$

Can suppose $|h|_{C^0} = h(x_0)$, $x_0 \geq 0$.

Then for suitable B (actually $B_i \to B$, $B_i \in \mathbb{Q}$), $\mu_{2,B}(\varphi_F^1) = |h|_{C^0}$ (Lagr. control).

But $\mu_{2,B}(\varphi_F^1) \leq d_{\mathrm{Hofer}}(\varphi_F^1, id) \leq |h|_{C^0}$ (Hofer-Lipschitz), with $\mu_{2,B}(\varphi_F^1) = |h|_{C^0}$. ∎

$$\begin{array}{ccc} \mu_{2,B}(\varphi_F^1) & \leq & d_{\mathrm{Hofer}}(\varphi_F^1, id) \\ \| & & \wedge\!\!\wedge \\ |h|_{C^0} & & |h|_{C^0} \end{array}$$

stabilization

Thm D: similar, but to prove them on the level of $\mathrm{Ham}(M_a)$ want to treat the effect of Gromov's exotic loop in $\pi_1 \mathrm{Ham}(M_a)$.

small geometric scale embeddings

Thm B: similar, but use $\mu_{k,B}$ for large enough k and conjugation invariance to put a cap of area $\frac{1}{k}$ inside $\mathcal{U}$.

conjugation invariance to put a cap of area k inside $\mathcal{U}$.

Packing

$\underline{\text{Thm}}^E$: Set $\quad \bar{d}(\varphi) = \lim_{m\to\infty} \frac{1}{m} \tilde{d}(\varphi^m, id)$.

$\varphi \in \widetilde{Ham}(M_a)$

By homogeneity and Hofer-Lipschitz property

$$(*) \quad \mu_{k,B}(\varphi) \le \bar{d}_{Hofer}(\varphi)$$

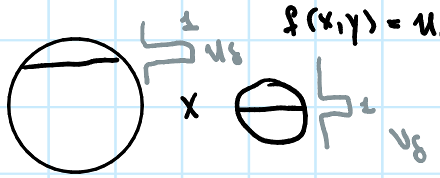
Consider $L = z^{-1}(\frac{1}{2} - B) = \partial D$, area$(D) = B$,

$\frac{1}{k+1} < B < \frac{1}{k}$.

and $\Lambda = L \times S^1$.

$z: S^2 \to [-\frac{1}{2}, \frac{1}{2}]$

Let f = smoothing of char. function of Λ.



$f(x,y) = u_\delta(x) v_\delta(y)$

$f: M_a \to [0,1]$

$f|_\Lambda \equiv 1$.

δ- arbitrarily small.

Modify to have zero mean keeping $|f|_{C^0} = 1$, near $\mathbb{L}_{k,B}$ but in $M_a \setminus \mathbb{L}_{k,B}$.

1. We have $\boxed{\bar{d}(\varphi_f) \geq \mu_{k,B}(\varphi_f) = \frac{1}{k}}$

Lagrangian control.

2. If Λ can be $\ell = k+1$ packed,

Sikorav's trick (for δ suff. small)

$\rightsquigarrow \boxed{\bar{d}(\varphi_f) \le \frac{1}{\ell}}$

Cannot have. ∎

$(-, id) \le t$

Cannot have. ∎

$$d_{Hofer}(\varphi_f^{\ell t}, \overbrace{\varphi_{f_1}^t \cdots \varphi_{f_\ell}^t}^{d_{Hofer}(-,\, id) \le t}) \le const_\ell$$

Indeed $d_{Hofer}(\varphi_f^{\ell t}, \varphi_{f_1}^t \cdots \varphi_{f_\ell}^t) \le const_\ell$

& $|f_1 + \dots + f_\ell|_{C^0} = 1$ supports of f_j disjoint.

$\Rightarrow \frac{1}{\ell t} d(\varphi_f^{\ell t}, id) \le \frac{1}{\ell t}(t + const_\ell) = \frac{1}{\ell} + O(\frac{1}{t})$

Technical overview:

1. Prove that no Maslov 0 $v: \Sigma \to M_a$, $\Sigma \xrightarrow{\pi} D$, contributes to suitable counts.

2. Dimension formula for $u: D \to Sym^k(M_a)$ with
   - ℓ_1 interior marked points constrained to a divisor $= \times\gamma$
   - ℓ_2 orbifold points constrained to $\Delta_{M_a} \times Sym^{k-2}(M_a)$
   - r boundary inputs constrained to $b \in H^1(\mathcal{L}_{K,B})$
   - 1 boundary output.

conj. class of transpositions in Sym_k

$$\text{v.dim} = \dim \mathcal{L}_{K,B} + (2-n)(k - \chi(\Sigma)) + \mu(v) - 2$$

$$\chi(\Sigma) = k - \ell_2$$

$$\dim \mathcal{L}_{K,B} = nk$$

$n=2$, $\mu(v) \ge 0$

+ geometry of problem $\Rightarrow$ $\mu(v) = 2$ if vdim $= \dim \mathcal{L}_{K,B}$

and all $b \in H^1(\mathcal{L}_{K,B})$ are weak bounding cochains.

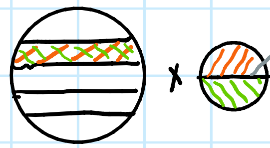
weak bounding cochains.

③ Prove that superpotential $W^{\mathfrak{b}}(b)$ has crit. pts b for $\mathfrak{b} = \mathfrak{b}_{smooth} + \lambda [x_\gamma]$, by studying smallest area terms

$\lambda^2 = c \cdot T^{B-C-a}$, $c \in \mathbb{C}$. Get $HF((L_{x,\mathfrak{b}}, b, b), (L_{x,\mathfrak{b}}, b, b)) \neq 0$.

① disks in S^2 & in $S^2(2a)$ B a

② annuli $C+a$



just 2 families & for each pair of consec. curves (rest ruled out by Mak-Smith)

①

Technical features: general non-resonance condition for cx. str. given by Fukaya trick to rule out Maslov 0 curves.

Follows from the matrix of periods of harmonic conjugates to harmonic measures on RS with bdry being symmetric (Ahlfors). Non-existence of RS surjecting to two annuli with bdries on two "non-resonant" sets of circles

② Showing existence of critical points of superpotential reduces to A_k Cartan matrix

$$\begin{pmatrix} 2 & -1 & 0 & & 0 \\ -1 & 2 & -1 & & \\ & & \ddots & & \\ 0 & & -1 & 2 & -1 \\ & & & -1 & 2 \end{pmatrix}$$

being invertible.

③ Rationality of α, B important for speciality property of $C(\mathcal{L}_{k,B}, 1; H)$ which is used for proving eg. Lagr. control.

④ Thm^C on C^0-topology uses $c_{k,B}$ directly. NOT $\mu_{k,B}$

Not[4] homogeneous, not[3] conj. invariant but[4'] subadditive and [3'] additive for φ_H, φ for H supported in $[0,1]\times U$, $U \cap \underline{\mathbb{L}}_{k,B} = \phi$.

Look at $\tau_{k,B} = c_{k,B} - \boxed{c_{1,\frac{1}{2}}}$ quasi-morphism

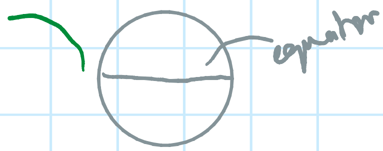
C^0-continuous, Hofer-Lipschitz, subadditive up to finite error.

Idea: Look at $\tau_{k,B}(H(\frac{1}{2}-z))$

$H \in C^0([0,\frac{1}{2}])$

$H(\frac{1}{2})=0$, extend by 0

$\xrightarrow{k\to\infty} \int_B^{\frac{1}{2}} H(s)\,ds$



So $\varphi^1_{H(\frac{1}{2}-z)} \in G^F \Rightarrow \int H \in C^1([0,\frac{1}{2}])$ bounded

primitive functions

& use quasi-subadditivity.

⑤ PFH and LOFH: working on S^2 can take $B_0 = C_0 = \frac{1}{k+1}$. limiting case

Expect relation of μ_{k,B_0} and PFH counterparts:

~ agree on $H(z)$.

~ Heegaard-Floer

~ Symmetric products

Thank you!