Variation of FLTZ skeleta

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Outline

In this talk, we introduce a particular “noncharacteristic” family of skeleta in T^*T^n, inspired by studies of derived coherent sheaf categories of GIT quotients [Halpern-Leistner and Sam ’16].

ArXiv:2011.06114, joint work with Peng Zhou.

(Local) skeleta of interest

Definition (rectilinear skeleton)

Let $[n] = \{1, \ldots, n\}$. A *rectilinear skeleton* is a skeleton

$$\Lambda_{\mathcal{I}} := \bigcup_{I \in \mathcal{I}} ss(\mathbb{C}_{P_I}) \subset T^*\mathbb{R}^n \tag{1}$$

where $\mathcal{I} \subset 2^{[n]}$ is any poset, where $P_I = \begin{cases} x_i \text{ free} & i \in I \\ x_i > 0 & i \notin I \end{cases}$

(Local) skeleta of interest

Definition (local quasiaffine FLTZ skeleton)

Let $\mathcal{I}$ be a poset such that $\Sigma_{\mathcal{I}} = \{\sigma_{I^c} : I \in \mathcal{I}\}$ forms a fan.

$$\begin{aligned} \Lambda_{\mathcal{I}}^{FLTZ} &:= \bigcup_{I \in \mathcal{I}} \mathbb{R}^I \times (-\sigma_{I^c}) \subset T^*\mathbb{R}^n \\ &= \Lambda_{\mathcal{I}} \end{aligned} \tag{2}$$

EX: $n=2$. only five possibilities

Upshot

- When $\mathcal{I}$ forms a fan $\Sigma_{\mathcal{I}}$, $\Lambda_{\mathcal{I}}$ is a local picture of the $\mathbb{Z}^n$-equivariant FLTZ skeleton, mirror to $[X_{\Sigma_{\mathcal{I}}}/T^n_{\mathbb{C}}]$.
- Sometimes, $\Lambda_{\mathcal{I}}$ forms a nice family of skeleta in $T^*\mathbb{R}^{n-k}$ under a linear map $\mathbb{R}^n \to \mathbb{R}^k$, with equivalent $Sh^\diamond$ on the fiber.
- We use this phenomenon to geometrically interpolate non-equivariant FLTZ skeleta mirror to a class of derived equivalent (non-compact) CY toric varieties under VGIT.
- Idea comes from window categories in GIT, constructed from a zonotope in $\mathbb{R}^k$ under certain "quasi-symmetric" condition.

Setting

Definition (quasi-symmetry)

Let $\mu : \mathbb{R}^n \to \mathbb{R}^k$ be a surjective linear map induced from a lattice map. Let $\beta_i := \mu(e_i)$. We say μ is

- *CY* if $\sum_i \mu(e_i) = 0$
- *quasi-symmetric* if $\sum_{\beta_i \in L} \beta_i = 0$ along each line L through the origin. Equivalently, there is a factorization of μ into

$$\mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^m \xrightarrow{q} \mathbb{R}^k \tag{3}$$

  where $m = \#$ of lines containing some β_i.

Key object

Consider zonotope $\nabla := \frac{1}{2}\mu[0,1]^n$ and generic shift $\delta \in \mathbb{R}^k$ s.t. $\partial(\nabla + \delta) \cap \mathbb{Z}^k = \emptyset$. Call $W_\delta := (\nabla + \delta) \cap \mathbb{Z}^k$ a *window*.

Zonotope window

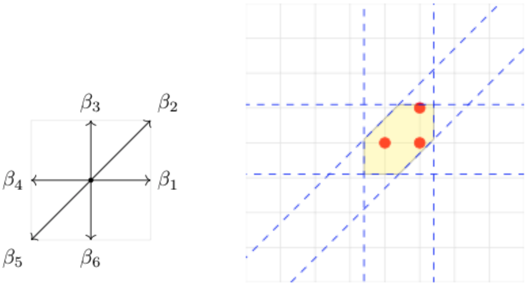


Figure 1: β_i distribution in $\mathbb{Z}^2$ and zonotope

A word on GIT

- β_i=weights of $T^k_{\mathbb{C}}$-action on $\mathbb{C}^n$.
- GIT parameter $I \in \mathbb{R}^k = Hom_{ab}(T^k_{\mathbb{C}}, \mathbb{Z}) \otimes \mathbb{R}$.

Theorem (Halpern-Leistner-Sam, 2016)

$M_\delta :=$ subcategory of $D^b_{coh}([\mathbb{C}^n / T^k_{\mathbb{C}}])$ generated by $O(\chi)$, $\chi \in (\nabla + \delta) \cap \mathbb{Z}^k$. $M_\delta \simeq D^b_{coh}([\mathbb{C}^n //_I T^k_{\mathbb{C}}])$ upon applying i^*.

Theorem (Kite, 2018)

In the quasi-symmetric case, GKZ fan is given by $\mathcal{H}_\nabla =$ linear hyperplane arrangement parallel to facets of ∇.

Idea: build skeleton over $\mathbb{R}^k$. Fiber skeleton lives in $T^*(\mathbb{R}^k / M) \simeq T^* T^{n-k}$ where $M := \ker \mu_{\mathbb{Z}}$. Over the chamber, get semistable (FLTZ) skeleton mirror to $[\mathbb{C}^n //_I T^k_{\mathbb{C}}]$.

Window skeleton: construction

Definition (Window skeleton)

$$\Lambda_\delta := \bigcup_{\tilde{v}\in\mu^{-1}(W_\delta)} ss(\mathbb{C}_{\tilde{v}+\mathbb{R}^n_{>0}})/M \subset T^*(\mathbb{R}^n/M) \tag{4}$$

Theorem (Zhou-H., 2020)

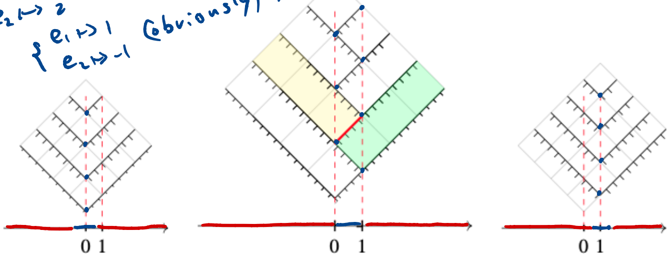
Let $\mathbb{H}_{\nabla,\delta} := \mathcal{H}_\nabla + \nabla + \delta$ thick walls. Λ_δ coincides with the semistable skeleton over each chamber complement to $\mathbb{H}_{\nabla,\delta}$.

Proof: Not too hard. Inspect local rectilinear skeleta using characterization of stability and properties of ∇.

thick walls = "trasit plaza" for FLTZ semistable skeleta

Simple example

* correction: in the video recording I realized I said $\begin{cases} e_1 \mapsto 1 \\ e_2 \mapsto 2 \end{cases}$ by mistake. It should be $\begin{cases} e_1 \mapsto 1 \\ e_2 \mapsto -1 \end{cases}$ (obviously) for a linear map.



$\delta = -\frac{1}{2}$ $\delta = 0$ (characteristic) $\delta = \frac{1}{2}$

Figure 2: window skeleton for $\mathbb{C}^*$ action $\lambda \cdot (z_1, z_2) = (\lambda z_1, \lambda^{-1} z_2)$ on $\mathbb{C}^2$

Hexagon example revisited

Base of $\Lambda_\delta \subset T^*(\mathbb{R}^6/\mathbb{Z}^4)$ is $\mathbb{R}^2$ decomposed into thick walls and chambers.

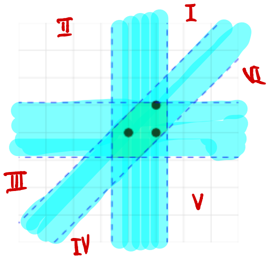


Figure 3: Thick walls and chambers

Window skeleton: main theorem

Theorem (Zhou-H., 2020)

For any $l \in \mathbb{R}^k$*, restriction to the fiber* $F_l = \mu_{\mathbb{R}}^{-1}(l)/M$ *induces an equivalence of categories*

$$\rho_{\delta,l} : Sh^{\diamond}(\mathbb{R}^n/M, \Lambda_\delta) \xrightarrow{\simeq} Sh^{\diamond}(F_l, \Lambda_\delta|_l). \tag{5}$$

Window objects generate $Sh^{\diamond}(\mathbb{R}^n/M, \Lambda_\delta)$. $\Rightarrow$
$Sh^{\diamond}(\mathbb{R}^n/M, \Lambda_\delta) \simeq M_\delta \Rightarrow$ *get universal HMS for generic quasi-symmetric GIT quotients.*

Proof: Much harder.

Summary of proof of nc deformation

- View $Sh^{\diamond}_{\Lambda_\delta}$ as a sheaf of categories. Equivalent to show restriction and corestriction (left-adjoint) are both fully-faithful.
- Inspect restriction and corestriction functors for $Sh^{\diamond}_{\Lambda_\delta}$ with respect to local rectilinear skeleta near lattice points in the universal cover.
- $F_I :=$ stalk corepresentative for $Q_I = \begin{cases} x_i < 0 & i \in I \\ x_i > 0 & i \notin I \end{cases}$, $\exists$ huge resolution of F_I by P_I, using order complex of $\mathcal{I}$.
- Restriction: for arbitrary δ, obstruction $=$ interior conormal of the affine hull of faces ∇ containing points in $\mathbb{Z}^k$.

Summary of proof of nc deformation cont'd

- Corestriction (hard): Recall that $\mathbb{R}^n \xrightarrow{\pi} \mathbb{R}^m \xrightarrow{q} \mathbb{R}^k$.
- Key idea: simplify the resolution of F_I in terms of intersection patterns between $q^{-1}(\nabla + \delta) + \pi(e_I) + \pi(\mathbb{R}^I_{\geq 0})$ and thickened coordinate cones in $\mathbb{R}^m$ and then describe support of F_I away from Q_I (=: "Flood(I)").
- Relate Flood(I) to the faces of Q_I where skeleton is absent from $\mathcal{I} = 2^{[n]}$ (=: "Leak(I)").
  $\overline{\pi(Q_I)} \cap \text{Flood}(I) = \overline{\pi(Q_I)} \cap \text{Leak}(I) \Rightarrow$ corestriction is fully-faithful.

Thank you!