SU(n) Casson invariants and symplectic geometry

Shaoyun Bai
(Princeton University)

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- Here, an SU(2) representation is called *irreducible*, if the commutator of its image is equal to the center of SU(2) (namely $\{\pm 1\}$). Otherwise, it is called *reducible*.
- Many topological applications, e.g. existence of non-triangulable 4–manifolds.

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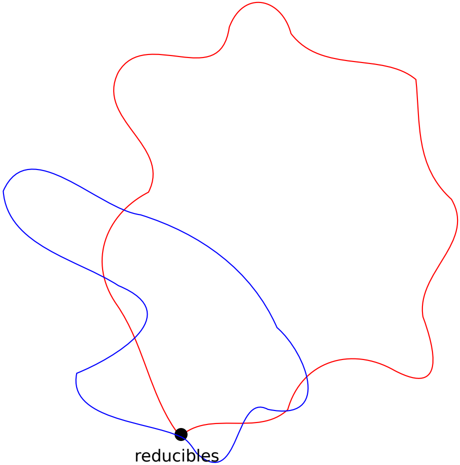
- The SU(2) character variety (i.e. the space of SU(2) representations up to conjugations) of $\pi_1(H_i)$ can be viewed as subsets of of the character variety of $\pi_1(\Sigma)$.
- The character variety of $\pi_1(Y)$ is equal to the intersection of the character varieties of $\pi_1(H_1)$ and $\pi_1(H_2)$ in the character variety of $\pi_1(\Sigma)$.

The construction of Casson invariant



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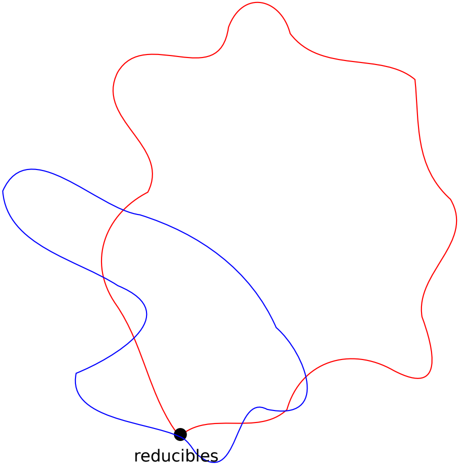

- It turns out that the intersection number is always even. The Casson invariant $\lambda(Y)$ is defined to be $1/2$ times the intersection number.

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- Reducible representations are no longer isolated for $n \geq 3$;
- The character varieties have singular points, hard to make perturbations;
- Even if transversality is achieved, the naive definition of intersection number depends on the perturbation.

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- 3-dimensional gauge-theoretic construction: Taubes (1990) for SU(2), Boden-Herald (1998) for SU(3).

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- Cappell-Lee-Miller (1990),
- Curtis (1994) $[\mathrm{SO}(3), \mathrm{U}(2), \mathrm{Spin}(4), \mathrm{SO}(4)]$.
- 3-dimensional gauge-theoretic construction: Taubes (1990) for SU(2), Boden-Herald (1998) for SU(3).
- Moreover, Taubes shows that the gauge-theoretic definition equals Casson's original intersection-theoretic definition.

Main result

Construction (B-Zhang, 2020)

Generalization of the definition of the Casson invariant to $\mathrm{SU}(n)$ using gauge theory.

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The $SU(n)$ Casson invariant is equal to a version of equivariant intersection number of character varieties.

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Theorem (B, 2021)

The $SU(n)$ Casson invariant is equal to a version of equivariant intersection number of character varieties.

The above theorem extends Taubes' result to all $SU(n)$: equivariant decategorified Atiyah-Floer conjecture.

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- G–orbits of $L_1 \cap L_2$ have $1-1$ correspondence with conjugacy classes of representations from $\pi_1(Y)$ to G.
- We will study equivariant geometry instead of orbifolds.

Equivariant Lagrangians and transversality

Definition

We say that L_1 and L_2 intersect *non-degenerately*, if L_1 and L_2 *have clean intersection along* $\mathrm{Orb}(p)$ *for each* $p \in L_1 \cap L_2$.

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- One can also analyze wall-crossings of the intersection when deforming the Hamiltonian perturbation in 1–parameter family.
- For the symplectic manifold $\mathcal{M}^{\mathfrak{g}}(\Sigma)$, the Hamiltonian perturbations could be related to holonomy perturbations on Y.
- We have a correspondence: perturbed intersections of L_1 and $L_2 \Leftrightarrow$ perturbed flat connections on Y.

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Maslov index and spectral flow

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- Amusingly again, many ingredients date back to the 1990s.

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- The above identification theorem between spectral flows and Maslov indices identifies the equivariant intersection number with our earlier gauge-theoretic definition.
- To show it is a topological invariant of Y, it suffices to show that the gauge-theoretic definition is an invariant.

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- This is done in the earlier work with Zhang.

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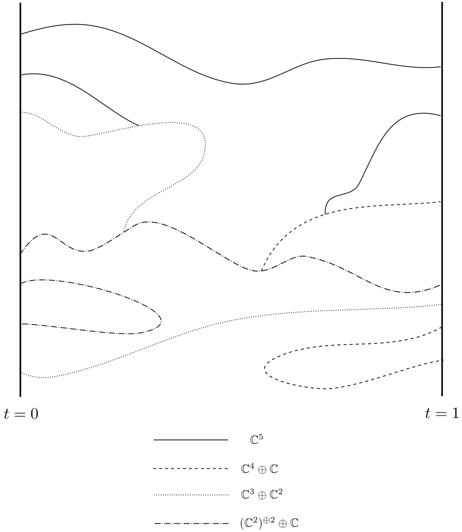

SU(3)–Casson invariant

$$\lambda_{\mathrm{SU}(3)}(Y) = \sum_{[p]\in(\Phi_H(L_1)\cap L_2)^{irr}} (-1)^{\mu(D(p))}$$

$$- \sum_{[p]\in(\Phi_H(L_1)\cap L_2)^{red}} (-1)^{\mu_t(D(p))}\left(\mu_n(D(p)) - \frac{\omega(D(\hat{p}))}{2\pi^2} + 1\right)$$

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This shows that Boden-Herald's SU(3) Casson invariant is the natural generalization of Walker's invariant for rational homology spheres.

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Further questions

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Conjecture

Suppose $K \subset Y$ is a knot and let $Y_{1/k}$ be the 3–manifold obtained from Y by doing $1/k$–Dehn surgery along K. Then $\lambda_{\mathrm{SU}(3)}(Y_{1/k}) = O(k^2)$ as $k \to \infty$ and $\lim_{k\to\infty} \frac{\lambda_{\mathrm{SU}(3)}(Y_{1/k})}{k^2}$ recovers the SU(3) version of Casson-Lin type invariant.

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- An extension of the weighted counting discussed above in the setting of J–holomorphic curves should be related to a symplectic definition of Gopakumar-Vafa invariants.

Thanks!