The large scale geometry of overtwisted contact forms

Thomas Melistas

University of Georgia

Symplectic Zoominar
March 26, 2021

Outline

Main Result

- It allows us to understand the large scale geometry of the space of contact forms supporting an overtwisted contact structure ξ_{ot} on a closed contact manifold Y.

Main Result

- It allows us to understand the large scale geometry of the space of contact forms supporting an overtwisted contact structure ξ_{ot} on a closed contact manifold Y.
- Let $\mathbb{H}$ denote the lower half-space in $\mathbb{R}^2$ and d_∞ denote the metric induced from the norm $|| \cdot ||_\infty$ in $\mathbb{R}^2$.

Main Result

- It allows us to understand the large scale geometry of the space of contact forms supporting an overtwisted contact structure ξ_{ot} on a closed contact manifold Y.
- Let $\mathbb{H}$ denote the lower half-space in $\mathbb{R}^2$ and d_∞ denote the metric induced from the norm $||\cdot||_\infty$ in $\mathbb{R}^2$.

Theorem

There exists a bi-Lipschitz embedding $F : (\mathbb{H}, d_\infty) \to (\mathcal{C}^Y_{\xi_{ot}}, d_{CBM})$

Main Result

- It allows us to understand the large scale geometry of the space of contact forms supporting an overtwisted contact structure ξ_{ot} on a closed contact manifold Y.
- Let $\mathbb{H}$ denote the lower half-space in $\mathbb{R}^2$ and d_∞ denote the metric induced from the norm $||\cdot||_\infty$ in $\mathbb{R}^2$.

Theorem

There exists a bi-Lipschitz embedding $F : (\mathbb{H}, d_\infty) \to (\mathcal{C}^Y_{\xi_{ot}}, d_{CBM})$

Definition

A map $f : (M_1, d_1) \to (M_2, d_2)$ is called a quasi-isometry if there exist $A \geq 1, B \geq 0, C \geq 0$ such that $\forall x, y \in M_1$

$$\frac{1}{A} d_1(x,y) - B \leq d_2(f(x), f(y)) \leq A d_1(x,y) + B$$

and

$$\forall z \in M_2, \quad \exists x \in M_1, \quad \text{so that} \quad d_2(z, f(x)) \leq C$$

A distance originating from Convex Geometry

Definition

Let K, L be convex bodies in $\mathbb{R}^n$. The Banach-Mazur distance between K and L is

$$d_{BM}(K,L) := \inf\left\{ a \geq 1 \,\middle|\, \begin{array}{l} \exists\ T \in GL(n), v, w \in \mathbb{R}^n \\ \frac{1}{a}(L+v) \subseteq T(K+w) \subseteq a(L+v) \end{array} \right\}$$

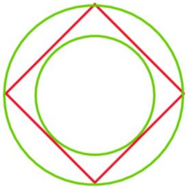

Figure: The linear Banach-Mazur distance

A distance originating from Convex Geometry

Definition

Let K, L be convex bodies in $\mathbb{R}^n$. The Banach-Mazur distance between K and L is

$$d_{BM}(K, L) := \inf \left\{ a \geq 1 \;\middle|\; \begin{array}{l} \exists\ T \in GL(n), v, w \in \mathbb{R}^n \\ \frac{1}{a}(L+v) \subseteq T(K+w) \subseteq a(L+v) \end{array} \right\}$$

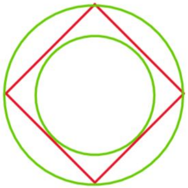

Figure: The linear Banach-Mazur distance

- Ostrover and Polterovich inspired by this proposed an analogous distance in the symplectic geometry setting.
- Usher, Gutt, Zhang and Stojisavljiević developed it further.

The definition of d_{CBM}

Definition

By a cs-embedding of a strict contact manifold (Y, α) to $(SY, d\theta)$ we mean an embedding $\phi : (Y, \alpha) \to (SY, d\theta)$ with $\phi^*(\theta + \eta) = \alpha$, where η is an exact, compactly supported 1-form on SY.

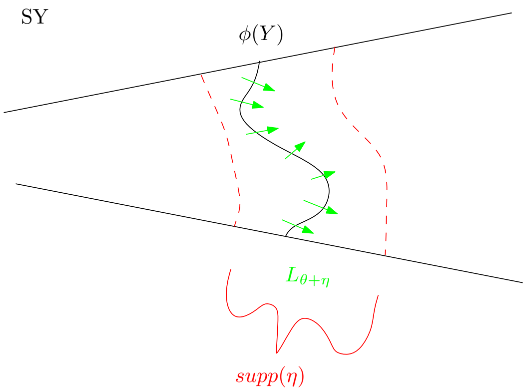

Figure: A cs-embedding.

The definition of d_{CBM}

Definition

We define $W(\beta) = \{p \in SY \mid 0 < p(v) \leq \beta(v), \forall v \in TY \text{ such that } \beta(v) > 0\}$.

The definition of d_{CBM}

Definition

We define $W(\beta) = \{p \in SY \mid 0 < p(v) \leq \beta(v), \forall v \in TY \text{ such that } \beta(v) > 0\}$.

Definition

$\alpha \prec \beta$ iff there is a cs-embedding $\phi : (Y, \alpha) \to SY$ such that $\phi(Y) \subset W(\beta)$.

The definition of d_{CBM}

Definition

We define $W(\beta) = \{p \in SY \mid 0 < p(v) \leq \beta(v), \forall v \in TY \text{ such that } \beta(v) > 0\}$.

Definition

$\alpha \prec \beta$ iff there is a cs-embedding $\phi : (Y, \alpha) \to SY$ such that $\phi(Y) \subset W(\beta)$.

Definition

Let (Y, α), (Y, β) be two contact manifolds in the same contactomorphism class and $(SY, d\theta)$ their common symplectization. We define the contact Banach-Mazur distance between α and β to be

$$d_{CBM}(\alpha, \beta) := \inf\{\ln(C) \in [0, \infty) \mid \alpha \prec C \cdot \beta, \beta \prec C \cdot \alpha\}$$

Contact Homology

- Contact Homology is a homology theory with generators monomials in good Reeb orbits and differential counting certain pseudoholomorphic curves in the symplectization. [Pardon, 2019].

Contact Homology

- Contact Homology is a homology theory with generators monomials in good Reeb orbits and differential counting certain pseudoholomorphic curves in the symplectization. [Pardon, 2019].
- It is an invariant of the contact structure and not of the chosen form used to define it.
- It was shown in [Yau, 2004] that contact homology of overtwisted contact structures vanishes.

Contact Homology

- Contact Homology is a homology theory with generators monomials in good Reeb orbits and differential counting certain pseudoholomorphic curves in the symplectization. [Pardon, 2019].
- It is an invariant of the contact structure and not of the chosen form used to define it.
- It was shown in [Yau, 2004] that contact homology of overtwisted contact structures vanishes.
- So it appears that we are not able to extract any meaningful information just by looking at contact homology itself.

Contact Homology

- Contact Homology is a homology theory with generators monomials in good Reeb orbits and differential counting certain pseudoholomorphic curves in the symplectization. [Pardon, 2019].
- It is an invariant of the contact structure and not of the chosen form used to define it.
- It was shown in [Yau, 2004] that contact homology of overtwisted contact structures vanishes.
- So it appears that we are not able to extract any meaningful information just by looking at contact homology itself.

Remark

There is a **filtration** by the action of Reeb orbits. The filtered version $CH^{\leq l}(M, \alpha)$ is well defined as the differential decreases action.

Contact Homology

- Contact Homology is a homology theory with generators monomials in good Reeb orbits and differential counting certain pseudoholomorphic curves in the symplectization. [Pardon, 2019].
- It is an invariant of the contact structure and not of the chosen form used to define it.
- It was shown in [Yau, 2004] that contact homology of overtwisted contact structures vanishes.
- So it appears that we are not able to extract any meaningful information just by looking at contact homology itself.

Remark

There is a **filtration** by the action of Reeb orbits. The filtered version $CH^{\leq l}(M, \alpha)$ is well defined as the differential decreases action.

- Furthermore, the filtration is sensitive to the chosen contact form and hence the barcode of the persistence module $CH^{\leq l}(M, \alpha)$ potentially has meaningful information.

Looking at the barcode

- The fact that $CH(Y, \xi_{ot}) = 0$ implies that there are no infinite bars.
- The next observation picks out the most essential finite bar.

Looking at the barcode

- The fact that $CH(Y, \xi_{ot}) = 0$ implies that there are no infinite bars.
- The next observation picks out the most essential finite bar.

Remark

Assume $\partial x = 1$. Then for a closed element y, by Leibniz rule we have $\partial(xy) = (\partial x)y \pm x(\partial y) = y$. This shows

- Exactness of the identity is enough for the homology to vanish.
- $\mathcal{A}(x \cdot y) = \mathcal{A}(x) + \mathcal{A}(y)$. Hence, the vanishing level of the class $[y]$ is at most $\mathcal{A}(x) + \mathcal{A}(y)$ which shows that the length of the bar corresponding to $[y]$ is at most $\mathcal{A}(x) + \mathcal{A}(y) - \mathcal{A}(y) = \mathcal{A}(x)$

Looking at the barcode

- The fact that $CH(Y, \xi_{ot}) = 0$ implies that there are no infinite bars.
- The next observation picks out the most essential finite bar.

Remark

Assume $\partial x = 1$. Then for a closed element y, by Leibniz rule we have $\partial(xy) = (\partial x)y \pm x(\partial y) = y$. This shows

- Exactness of the identity is enough for the homology to vanish.
- $\mathcal{A}(x \cdot y) = \mathcal{A}(x) + \mathcal{A}(y)$. Hence, the vanishing level of the class $[y]$ is at most $\mathcal{A}(x) + \mathcal{A}(y)$ which shows that the length of the bar corresponding to $[y]$ is at most $\mathcal{A}(x) + \mathcal{A}(y) - \mathcal{A}(y) = \mathcal{A}(x)$

Definition

We define the l**-invariant** of an overtwisted contact form α_{ot} to be the action level for which the unit of the algebra becomes exact.

Looking at the barcode

- This remark reveals that the notion of boundary depth for Hamiltonian Floer homology introduced in [Usher, 2011] is the analogue to the l-invariant.

Looking at the barcode

- This remark reveals that the notion of boundary depth for Hamiltonian Floer homology introduced in [Usher, 2011] is the analogue to the l-invariant.
- Q: How can one control the l-invariant?

Looking at the barcode

- This remark reveals that the notion of boundary depth for Hamiltonian Floer homology introduced in [Usher, 2011] is the analogue to the l-invariant.
- Q: How can one control the l-invariant?
- A: Dynamics of Lutz twisting [Wendl, 2005].

Looking at the barcode

- This remark reveals that the notion of boundary depth for Hamiltonian Floer homology introduced in [Usher, 2011] is the analogue to the l-invariant.
- Q: How can one control the l-invariant?
- A: Dynamics of Lutz twisting [Wendl, 2005].
- Q: Is its modification Lipschitz?

Looking at the barcode

- This remark reveals that the notion of boundary depth for Hamiltonian Floer homology introduced in [Usher, 2011] is the analogue to the l-invariant.
- Q: How can one control the l-invariant?
- A: Dynamics of Lutz twisting [Wendl, 2005].
- Q: Is its modification Lipschitz?
- A: In general, boundary depth is Lipschitz, yet we will need to simultaneously control volume and the l-invariant. Computations involve Gray stability and compensating for the alteration of volume.

The Lutz twist

- A way to obtain an overtwisted contact structure starting with a tight one is the so called "Lutz twist".
- By a contact neighborhood theorem, in a neighborhood $S^1 \times D^2_\varepsilon$ of any transverse knot to the contact structure ξ, the contact structure is given by $\ker(d\theta + r^2 d\phi)$

The Lutz twist

- A way to obtain an overtwisted contact structure starting with a tight one is the so called "Lutz twist".
- By a contact neighborhood theorem, in a neighborhood $S^1 \times D^2_\varepsilon$ of any transverse knot to the contact structure ξ, the contact structure is given by $\ker(d\theta + r^2 d\phi)$

Definition

The (full) Lutz twist is the process of replacing the contact structure $\ker(d\theta + r^2 d\phi)$ by $\ker(h_1(r)d\theta + h_2(r)d\phi))$ where

- $h_1(r) = 1$ and $h_2(r) = r^2$, r near 0 and ε.
- $(h_1(r), h_2(r))$ is never parallel to $(h_1'(r), h_2'(r))$.
- The path determined by $(h_1(r), h_2(r))$ wraps once around the origin.

The path is visually understood from the following picture.

The Lutz twist

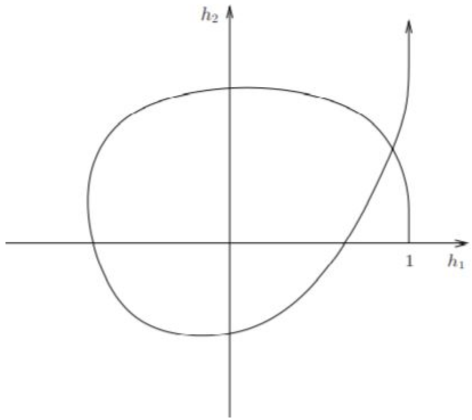

Figure: The path describing the full Lutz twist.

- For our construction though, we need the contact form and not only the contact structure to look like $d\theta + r^2 d\phi$ near the T^2 boundary of the contact neighborhood.
- This can be achieved by multiplying the original form by a smooth positive function f supported in a neighborhood of $T^2 = \partial(S^1 \times D^2_\varepsilon)$.

Sketch of proof

- The first degree of freedom in $\mathbb{R}^2$ is volume.
- The volume of a contact form is defined as $Vol(\alpha) := \int_Y \alpha \wedge d\alpha$.

Sketch of proof

- The first degree of freedom in $\mathbb{R}^2$ is volume.
- The volume of a contact form is defined as $Vol(\alpha) := \int_Y \alpha \wedge d\alpha$.
- The second degree is the l-invariant. This is the length of the largest finite bar and the action of the lowest action Reeb orbit bounding a unique pseudoholomorphic plane.

Sketch of proof

- The first degree of freedom in $\mathbb{R}^2$ is volume.
- The volume of a contact form is defined as $Vol(\alpha) := \int_Y \alpha \wedge d\alpha$.
- The second degree is the l-invariant. This is the length of the largest finite bar and the action of the lowest action Reeb orbit bounding a unique pseudoholomorphic plane.
- Note that the volume and l-invariant are quantities associated to each contact form, so we need to modify forms in a consistent way.

Sketch of proof

- The first degree of freedom in $\mathbb{R}^2$ is volume.
- The volume of a contact form is defined as $Vol(\alpha) := \int_Y \alpha \wedge d\alpha$.
- The second degree is the l-invariant. This is the length of the largest finite bar and the action of the lowest action Reeb orbit bounding a unique pseudoholomorphic plane.
- Note that the volume and l-invariant are quantities associated to each contact form, so we need to modify forms in a consistent way.
- The way to modify volume is multiplication of the original contact form by a constant.
- The way to modify the l-invariant is to perform a Lutz twist around a sufficiently small neighborhood of a transverse knot.

Sketch of proof

- The first degree of freedom in $\mathbb{R}^2$ is volume.
- The volume of a contact form is defined as $Vol(\alpha) := \int_Y \alpha \wedge d\alpha$.
- The second degree is the l-invariant. This is the length of the largest finite bar and the action of the lowest action Reeb orbit bounding a unique pseudoholomorphic plane.
- Note that the volume and l-invariant are quantities associated to each contact form, so we need to modify forms in a consistent way.
- The way to modify volume is multiplication of the original contact form by a constant.
- The way to modify the l-invariant is to perform a Lutz twist around a sufficiently small neighborhood of a transverse knot.
- Note that the two notions are not independent.

Sketch of proof

Preliminary modification

$$\alpha_{ot} = \begin{cases} h_1(r)d\theta + h_{2,l}(r)d\phi, & \text{on } S^1 \times D^2_\varepsilon \\ \alpha, & \text{otherwise} \end{cases}$$

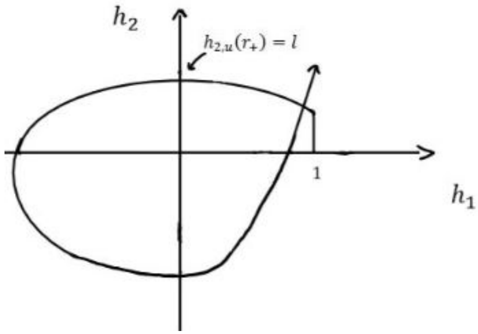

Figure: The path describing the 2-parameter family.

Sketch of proof

Preliminary modification

$$\alpha_{ot} = \begin{cases} h_1(r)d\theta + h_{2,l}(r)d\phi, & \text{on } S^1 \times D^2_\varepsilon \\ \alpha, & \text{otherwise} \end{cases}$$

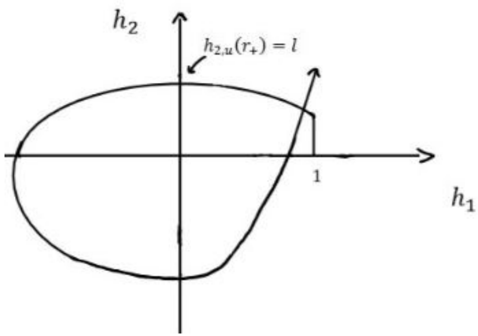

Figure: The path describing the 2-parameter family.

- We need a map that maps a pair $(\ln(\sqrt{k}), \ln(l))$ to a form $\alpha_{k,l}$ with $Vol(\alpha_{k,l}) = k$ and $l(\alpha_{k,l}) = l$.

Sketch of Proof

- This map is given by $(\ln(\sqrt{k}), \ln(l)) \mapsto \alpha_{k,l} = \sqrt{k} \cdot \alpha_{ot}$

Sketch of Proof

- This map is given by $(\ln(\sqrt{k}), \ln(l)) \mapsto \alpha_{k,l} = \sqrt{k} \cdot \alpha_{ot}$
- There is a way to modify l-invariant without impact on volume. (Compensating in a compatible way with compatible dynamics)

Sketch of Proof

- This map is given by $(\ln(\sqrt{k}), \ln(l)) \mapsto \alpha_{k,l} = \sqrt{k} \cdot \alpha_{ot}$
- There is a way to modify l-invariant without impact on volume. (Compensating in a compatible way with compatible dynamics)
- Changing volume changes the l-invariant.

Sketch of Proof

- This map is given by $(\ln(\sqrt{k}), \ln(l)) \mapsto \alpha_{k,l} = \sqrt{k} \cdot \alpha_{ot}$
- There is a way to modify l-invariant without impact on volume. (Compensating in a compatible way with compatible dynamics)
- Changing volume changes the l-invariant.
- A triangle inequality helps us overcome the dependence.

Sketch of Proof

- This map is given by $(\ln(\sqrt{k}), \ln(l)) \mapsto \alpha_{k,l} = \sqrt{k} \cdot \alpha_{ot}$
- There is a way to modify l-invariant without impact on volume. (Compensating in a compatible way with compatible dynamics)
- Changing volume changes the l-invariant.
- A triangle inequality helps us overcome the dependence.

The left inequality follows easily by the following lemma.

Lemma

- *If* $\alpha \prec \beta$, *then* $Vol(Y, \alpha) \leq Vol(Y, \beta)$
- $Vol(Y, C \cdot \alpha) = C^2 \cdot Vol(Y, \alpha)$
- *If* $\alpha \prec \beta$, *then* $l(\alpha) \leq l(\beta)$
- $l(C \cdot \alpha) = C \cdot l(\alpha)$

Sketch of proof

- For the right inequality we need the aforementioned triangle inequality.
- If we start from $(\ln(\sqrt{k_1}), \ln(l_1))$ and we need to end up to $(\ln(\sqrt{k_2}), \ln(l_2))$, then the intermediate point should be $\left(\ln(\sqrt{k_2}), \ln(\sqrt{\frac{k_2}{k_1}} l_1)\right)$.
- Calculations using Gray stability show that this association is Lipschitz!
- Explicitly,

$$\frac{1}{3} d_\infty((\ln(\sqrt{k_1}), \ln(l_1)), (\ln(\sqrt{k_2}), \ln(l_2)))$$
$$\leq d_{CBM}(\alpha_{k_1,l_1}, \alpha_{k_2,l_2}) \leq 3 d_\infty((\ln(\sqrt{k_1}), \ln(l_1)), (\ln(\sqrt{k_2}), \ln(l_2)))$$

Thank you!

Pardon, John (2019). “Contact homology and virtual fundamental cycles”. In: *Journal of the American Mathematical Society* 32.3, pp. 825–919.

Usher, Michael (2011). “Boundary depth in Floer theory and its applications to Hamiltonian dynamics and coisotropic submanifolds”. In: *Israel J. Math.* 184, pp. 1–57. ISSN: 0021-2172. DOI: 10.1007/s11856-011-0058-9. URL: https://doi.org/10.1007/s11856-011-0058-9.

Wendl, Chris (2005). “Finite energy foliations and surgery on transverse links”. PhD thesis. New York University, Graduate School of Arts and Science.

Yau, Mei-Lin (Oct. 2004). “Vanishing of the contact homology of overtwisted contact 3–manifolds”. In: *arXiv Mathematics e-prints*, math/0411014, math/0411014. arXiv: math/0411014 [math.SG].