The large scale geometry of overtwisted contact forms

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Main Result

- It allows us to understand the large scale geometry of the space of contact forms supporting an overtwisted contact structure $\xi_{ot}$ on a closed contact manifold $Y$. 
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**Theorem**

There exists a bi-Lipschitz embedding $F : (\mathbb{H}, d_\infty) \rightarrow (C^Y_{\xi_{ot}}, d_{CBM})$
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Definition

A map $f : (M_1, d_1) \to (M_2, d_2)$ is called a quasi-isometry if there exist $A \geq 1, B \geq 0, C \geq 0$ such that $\forall x, y \in M_1$

$$\frac{1}{A}d_1(x, y) - B \leq d_2(f(x), f(y)) \leq Ad_1(x, y) + B$$

and

$$\forall z \in M_2, \exists x \in M_1, \text{ so that } d_2(z, f(x)) \leq C$$
A distance originating from Convex Geometry

Definition

Let $K, L$ be convex bodies in $\mathbb{R}^n$. The Banach-Mazur distance between $K$ and $L$ is

$$d_{BM}(K, L) := \inf \left\{ a \geq 1 \left| \exists T \in GL(n), v, w \in \mathbb{R}^n \right. \frac{1}{a} (L + v) \subseteq T(K + w) \subseteq a(L + v) \right\}$$

Figure: The linear Banach-Mazur distance
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Figure: The linear Banach-Mazur distance

- Ostrover and Polterovich inspired by this proposed an analogous distance in the symplectic geometry setting.
- Usher, Gutt, Zhang and Stojisavljević developed it further.
The definition of $d_{CBM}$

**Definition**

By a cs-embedding of a strict contact manifold $(Y, \alpha)$ to $(SY, d\theta)$ we mean an embedding $\phi : (Y, \alpha) \rightarrow (SY, d\theta)$ with $\phi^*(\theta + \eta) = \alpha$, where $\eta$ is an exact, compactly supported 1-form on $SY$.

**Figure:** A cs-embedding.
The definition of $d_{CBM}$

**Definition**

We define $W(\beta) = \{ p \in SY | 0 < p(v) \leq \beta(v), \forall v \in TY \text{ such that } \beta(v) > 0 \}$. 
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**Definition**

$\alpha < \beta$ iff there is a cs-embedding $\phi : (Y, \alpha) \to SY$ such that $\phi(Y) \subset W(\beta)$. 
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We define $W(\beta) = \{ p \in SY \mid 0 < p(v) \leq \beta(v), \forall v \in TY \text{ such that } \beta(v) > 0 \}$.

**Definition**

$\alpha \prec \beta$ iff there is a cs-embedding $\phi : (Y, \alpha) \to SY$ such that $\phi(Y) \subset W(\beta)$.

**Definition**

Let $(Y, \alpha), (Y, \beta)$ be two contact manifolds in the same contactomorphism class and $(SY, d\theta)$ their common symplectization. We define the contact Banach-Mazur distance between $\alpha$ and $\beta$ to be

$$d_{CBM}(\alpha, \beta) := \inf\{ \ln(C') \in [0, \infty) \mid \alpha \prec C \cdot \beta, \beta \prec C \cdot \alpha \}$$
Contact Homology

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- So it appears that we are not able to extract any meaningful information just by looking at contact homology itself.

Remark

There is a filtration by the action of Reeb orbits. The filtered version \( C^H \leq \ell (M, \alpha) \) is well defined as the differential decreases action. Furthermore, the filtration is sensitive to the chosen contact form and hence the barcode of the persistence module \( C^H \leq \ell (M, \alpha) \) potentially has meaningful information.
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Looking at the barcode

- The fact that $CH(Y, \xi_{ot}) = 0$ implies that there are no infinite bars.
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**Remark**

Assume \( \partial x = 1 \). Then for a closed element \( y \), by Leibniz rule we have

\[
\partial(xy) = (\partial x)y \pm x(\partial y) = y.
\]

This shows

- Exactness of the identity is enough for the homology to vanish.
- \( A(x \cdot y) = A(x) + A(y) \). Hence, the vanishing level of the class \([y]\) is at most \( A(x) + A(y) \) which shows that the length of the bar corresponding to \([y]\) is at most \( A(x) + A(y) - A(y) = A(x) \).
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- Exactness of the identity is enough for the homology to vanish.
- $\mathcal{A}(x \cdot y) = \mathcal{A}(x) + \mathcal{A}(y)$. Hence, the vanishing level of the class $[y]$ is at most $\mathcal{A}(x) + \mathcal{A}(y)$ which shows that the length of the bar corresponding to $[y]$ is at most $\mathcal{A}(x) + \mathcal{A}(y) - \mathcal{A}(y) = \mathcal{A}(x)$

**Definition**

We define the **l-invariant** of an overtwisted contact form $\alpha_{ot}$ to be the action level for which the unit of the algebra becomes exact.
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Q: How can one control the $l$-invariant?

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A: In general, boundary depth is Lipschitz, yet we will need to simultaneously control volume and the $l$-invariant. Computations involve Gray stability and compensating for the alteration of volume.
The Lutz twist

- A way to obtain an overtwisted contact structure starting with a tight one is the so called “Lutz twist”.

- By a contact neighborhood theorem, in a neighborhood $S^1 \times D^2_{\epsilon}$ of any transverse knot to the contact structure $\xi$, the contact structure is given by $\ker(d\theta + r^2 d\phi)$

$h_1(r) = 1$ and $h_2(r) = r^2$, $r$ near $0$ and $\epsilon$.

$(h_1(r), h_2(r))$ is never parallel to $(h_1'(r), h_2'(r))$.

The path determined by $(h_1(r), h_2(r))$ wraps once around the origin.

The path is visually understood from the following picture.
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**Definition**

The (full) Lutz twist is the process of replacing the contact structure $\ker(d\theta + r^2 d\phi)$ by $\ker(h_1(r)d\theta + h_2(r)d\phi)$ where

- $h_1(r) = 1$ and $h_2(r) = r^2$, $r$ near 0 and $\varepsilon$.
- $(h_1(r), h_2(r))$ is never parallel to $(h'_1(r), h'_2(r))$.
- The path determined by $(h_1(r), h_2(r))$ wraps once around the origin.

The path is visually understood from the following picture.
The Lutz twist

Figure: The path describing the full Lutz twist.

- For our construction though, we need the contact form and not only the contact structure to look like $d\theta + r^2 d\phi$ near the $T^2$ boundary of the contact neighborhood.
- This can be achieved by multiplying the original form by a smooth positive function $f$ supported in a neighborhood of $T^2 = \partial(S^1 \times D^2_\varepsilon)$. 
Sketch of proof

- The first degree of freedom in $\mathbb{R}^2$ is volume.
- The volume of a contact form is defined as $Vol(\alpha) := \int_Y \alpha \wedge d\alpha$. 

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The second degree is the $l$-invariant. This is the length of the largest finite bar and the action of the lowest action Reeb orbit bounding a unique pseudoholomorphic plane.
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- Note that the two notions are not independent.
Sketch of proof

Preliminary modification

\[ \alpha_{ot} = \begin{cases} h_1(r) d\theta + h_{2,l}(r) d\phi, & \text{on } S^1 \times D^2_\varepsilon \\ \alpha, & \text{otherwise} \end{cases} \]

Figure: The path describing the 2-parameter family.
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Figure: The path describing the 2-parameter family.

- We need a map that maps a pair \((\ln(\sqrt{k}), \ln(l))\) to a form \(\alpha_{k,l}\) with \(Vol(\alpha_{k,l}) = k\) and \(l(\alpha_{k,l}) = l\).
This map is given by \((\ln(\sqrt{k}), \ln(l)) \mapsto \alpha_{k,l} = \sqrt{k} \cdot \alpha_{ot}\)
Sketch of Proof

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- A triangle inequality helps us overcome the dependence.

The left inequality follows easily by the following lemma.

Lemma

- If \(\alpha \prec \beta\), then \(Vol(Y, \alpha) \leq Vol(Y, \beta)\)
- \(Vol(Y, C \cdot \alpha) = C^2 \cdot Vol(Y, \alpha)\)
- If \(\alpha \prec \beta\), then \(l(\alpha) \leq l(\beta)\)
- \(l(C \cdot \alpha) = C \cdot l(\alpha)\)
Sketch of proof

- For the right inequality we need the aforementioned triangle inequality.
- If we start from \((\ln(\sqrt{k_1}), \ln(l_1))\) and we need to end up to \((\ln(\sqrt{k_2}), \ln(l_2))\), then the intermediate point should be \(\left(\ln(\sqrt{k_2}), \ln\left(\frac{k_2}{k_1} l_1\right)\right)\).
- Calculations using Gray stability show that this association is Lipschitz!
- Explicitly,

\[
\frac{1}{3} d_\infty((\ln(\sqrt{k_1}), \ln(l_1)), (\ln(\sqrt{k_2}), \ln(l_2))) \\
\leq d_{CBM}(\alpha_{k_1,l_1}, \alpha_{k_2,l_2}) \leq 3d_\infty((\ln(\sqrt{k_1}), \ln(l_1)), (\ln(\sqrt{k_2}), \ln(l_2)))
\]
Thank you!

