Categorical non-properness in wrapped Floer theory

Sheel Ganatra (USC)

Symplectic Zoominar, 4/2/2021.

Goal: discuss non-finiteness ('non-properness') phenomena frequently seen in wrapped Floer homology, and show that to a large part they are a necessary consequence of global categorical/TFT structures.

The phenomenon

(X^{2n}, λ) Liouville manifold ($\omega = d\lambda$ symplectic, s.t. flowing out by Z [$\iota_Z \omega = \lambda$] presents $X \setminus$ cpct subset as symplectization of contact manifold, $\partial_\infty X$.).

Assume $n > 0$ (so X is non-compact).

$\rightsquigarrow$ Symplectic cohomology $SH^\bullet(X)$. Can arrange $SH^\bullet(X) = H^\bullet\big(C^\bullet(X) \oplus C_\bullet(S^1)^{\#\text{ Reeb orbits}}, \delta\big)$

$\rightsquigarrow$ wrapped Fukaya category $\mathcal{W}(X)$

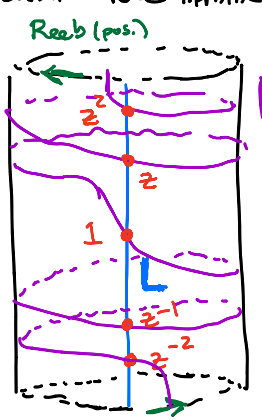
objects: $L \subseteq X$, exact Lagrangians, properly embedded, (possibly non-compact but) cylindrical @ ∞,

cohomological morphisms: $HW^\bullet(L_0, L_1) := \varinjlim_{\substack{L_0^+ \to L_0 \\ \text{positive at } \infty}} HF^\bullet(L_0^+, L)$.

can arrange $HW^\bullet(K, L) = H^\bullet\big(\underset{\substack{\| \\ C^\bullet(L) \text{ if } K = L}}{CF^\bullet(K, L)} \oplus \mathbb{K}^{\{\text{Reeb chords } \partial_\infty K \to \partial_\infty L\}}, \delta\big)$

The presence of chords/orbits in these complexes means that $SH^\bullet / HW^\bullet$ could be infinite dimensional, but doesn't force infinite dimensionality.

Ex: $X = T^*S^1$



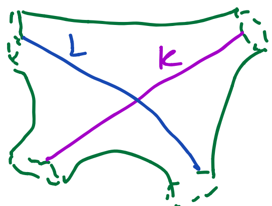
$HW^\bullet(L, L) = \varinjlim_i HF^\bullet(L^{(i)}, L) = \mathbb{C}[z, z^{-1}] \quad \big(= \mathrm{Ext}_{\mathbb{C}^*}(\mathcal{O}_{\mathbb{C}^*}, \mathcal{O}_{\mathbb{C}^*})\big).$

On the other hand, in all (to my knowledge) known explicit computations on Liouville manifolds ← numerous computations, all on Weinstein manifolds ($\exists$ gradient-like for a Morse exhaustion fxn.).

(a) $SH^\bullet(X) = 0$ or is infinite rank.

(b) for a non-compact connected L, $HW^\bullet(L,L) = 0$ or ∞.

Note: if K or L cpct, or more generally $\partial_\infty K$ and $\partial_\infty L$ lie on different components of $\partial_\infty X$, then $HW^\bullet(K,L) = HF^\bullet(K,L)$

Ex: 

$(HW(K,L) = \mathbb{K}.)$.

• $HW^*(K,L)$'s are nevertheless 'frequently' infinite.

Folk Q: Must these 'infinite or zero' phenomena hold?

Rmks:

• (a) is the special case $L = \Delta \subseteq X^- \times X$ of (b).

• unknown even for every $X = T^*Q$ that $SH^\bullet(X) \cong H_{n-\bullet}(\mathcal{L}Q)$ [Viterbo,.. Q spin for simplicity] is always infinite, to my understanding. (known if $\pi_1 Q = 0$ or Q nilpotent, also if $\pi_1 Q$ has ∞'ly many conj. classes)

• If (a) held, then Viterbo's acceleration map $H^\bullet(X) \xrightarrow{accel} SH^\bullet(X)$ (inclusion of a subcomplex in model above) cannot be an iso. ('alg. Weinstein conj.')

$\Rightarrow \exists$ a Reeb orbit on $\partial_\infty X$ (Weinstein conj.).

• On the other hand, [Ritter, Ritter-Smith]: on some negative line bundles, a class of monotone non-exact non-compact (contact at ∞) sympl. manifolds, $SH^\bullet(X)$ = finite non-zero, & similarly for $HW^\bullet$ (& $\mathcal{W}$ "smooth+proper").

It seems unclear/hard whether these hold. However, our main result is that a more 'global' version of the 'infinite or zero' phenomenon holds in broad generality (in the exact setting).:

means $\exists$ a collection of Lagrangians satisfying Abouzaid's generation criterion ($\mathcal{OC}: HH_{*-n}(\mathcal{W}(X)) \to SH^*(X)$ hits 1.)

Weinstein $\Rightarrow$ non-degenerate by work of [Chantraine-Dimitroglou-Rizell-Ghiggini-Golovko] [G.-Pardon-Shende], [G.], [Gao].

Thm ([G. in prep]): Let X be a non-degenerate Liouville manifold, e.g., any Weinstein manifold) of dim > 0.

Then:

(1) $\mathcal{W}(X)$ is either 0 or non-proper. ($\Rightarrow$ if $\mathcal{W}(X) \neq 0$ $\exists$ K, L with $HW^\bullet(K,L) = \infty$, including a pair in any (split)-generating set. $\Rightarrow$ if $\{\Delta_i\}$ split-generates, then $HW^\bullet(\sqcup \Delta_i, \sqcup \Delta_i) = 0$ or ∞)

(1') Any quotient of $\mathcal{W}(X)$ is 0 or non-proper. (excludes e.g., $\mathcal{W}(X) =$ non-proper $\oplus$ proper)

(2) Any $L \in \mathcal{W}(X)$ all of whose components are non-compact is either 0 or both left & right non-proper. ($\Rightarrow$ if $L \neq 0$, $\exists$ K, K' with $HW^\bullet(L,K) = \infty$, $HW^\bullet(K',L) = \infty$, including one in any (split)-generating set).

(2') Same for any summand K of L in perf $\mathcal{W}(X)$.

Rmk: Say $\mathcal{W}(X)$ gen. by $\{\Delta_i\}_{i=1}^k$ with each Δ_i non-compact (e.g., cocores of Weinstein pres. [CDGG, GPS]).

(Assume we've discarded all Δ_i which are zero). Then if $\mathcal{W}(X) \neq 0$, (1) $\Rightarrow$ some $HW^\bullet(\Delta_i, \Delta_j) = \infty$

$(2) \Rightarrow$ at least k of $\{HW^\bullet(\Delta_i, \Delta_j)\}_{i,j}$ are ∞, one for each i & j.

<u>Def</u>: $\mathbb{K}$ field, $\mathrm{Mod}\,\mathbb{K} :=$ chain cplxes$/\mathbb{K}$, $\mathrm{perf}\,\mathbb{K} :=$ chain complexes which are cohomologically finite $(\dim H^\bullet(C^\bullet) < \infty)$

- $\mathcal{C}$ is <u>proper</u> if $\hom_{\mathcal{C}}(X, Y) \in \mathrm{perf}\,\mathbb{K}$ $\forall\, X, Y \in \mathcal{C}$.

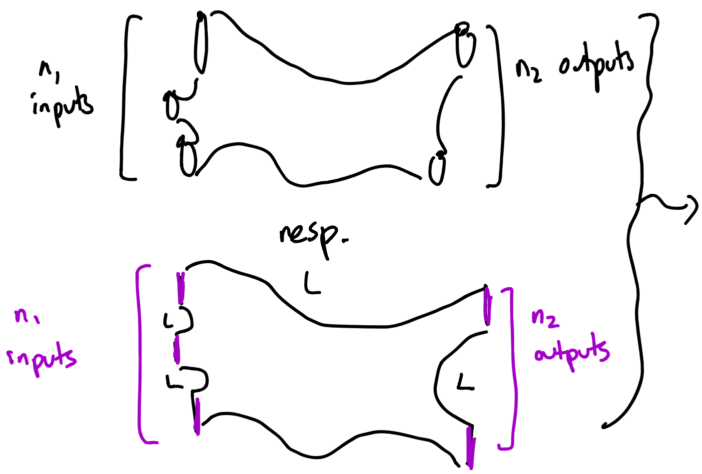
- $L \in \mathcal{C}$ is right (resp. left) proper if $\hom_{\mathcal{C}}(X, L)$ (resp. $\hom_{\mathcal{C}}(L, X)$) $\in \mathrm{perf}\,\mathbb{K}$ for all $X \in \mathcal{C}$.

'degenerate copairing'

The proofs appeal to the structure and properties of TQFT operations, and their interplay w/ the fact that $\mathcal{W}(X)$ is a smooth Calabi-Yau category [G.].

'global' statements: smoothness is stronger than existence of generators; & Calabi-Yau is a form of Poincaré duality for Hochschild invariants.

<u>TQFT structures</u>: Recall (c.f., [Ritter]) that $SH^\bullet(X)$ resp. $HW^\bullet(L,L)$ admit (closed resp. open) 2d TQFT operations, as is common in Floer theory; e.g., a pictures in genus 0.



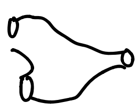
$$SH^\bullet(X)^{\otimes n_1} \to SH^\bullet(X)^{\otimes n_2}\,[(n_2+g-1)(2n)].$$

resp.

$$HW^\bullet(L,L)^{\otimes n_1} \to HW^\bullet(L,L)^{\otimes n_2}\,[n(n_2 - \chi(\Sigma))]$$

(<u>compatible with gluing surfaces</u>).

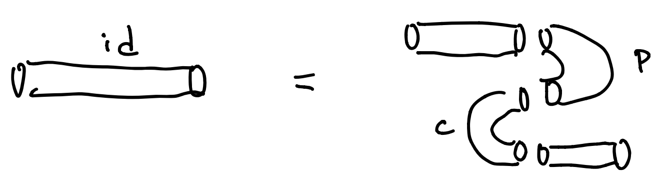
E.g., on $SH^\bullet(X)$:  $\longleftrightarrow$ id  $\longleftrightarrow$ unit

 $\longleftrightarrow$ product  $\longleftrightarrow$ $\boxed{\text{copairing}}$ $\rightsquigarrow$ $c_{SH} \in SH^\bullet(X)^{\otimes 2}[2n]$

$$c_{SH} = \sum_{\text{finite}} a_i \otimes b_i$$

There's an important restriction: $\boxed{n_2 = \#\text{outputs} \geq 1}$. $\exists$ technical problems ('failure' of maximum principle) in setting up a pairing, reflecting a more fundamental point: if $\exists$ a compatible pairing

$p = \langle -, - \rangle$:  : $(SH^\bullet)^{\otimes 2} \to \mathbb{K}$, then the $\boxed{\text{snake relation in TQFT}} \Rightarrow$



$$v = \mathrm{id}(v) = (p \otimes \mathrm{id})(v \otimes c) = \sum \langle v, a_i \rangle b_i.$$

$\Rightarrow SH^\bullet(X)$ $\boxed{\text{finite-dimensional}}$ and the maps $SH^\bullet(X) \xrightarrow{p^*} SH^\bullet(X)^*[-2n] \xrightarrow{c^*} SH^\bullet(X)$ are inverse $\Rightarrow c$ is non-degenerate.

$v \longmapsto \langle v, - \rangle$

$\phi \longmapsto \phi \otimes \mathrm{id}(c)$

So, having a pairing implies the copairing c is non-degenerate, & finite-dimensionality.

Idea 1: The failure of c to be non-degenerate (c^* to be an iso.) obstructs the existence of a pairing.

$\iff H^\bullet \mathrm{Cone}(c^*: SC^*(X)^\vee[-2n] \to SC^\bullet(X))$ non-zero.

n.b. identifying $SC^*(X)^\vee[-2n] \cong SC_\bullet(X)$, this is the Rabinowitz Floer homology $RFH^*(X)$ [Cieliebak-Frauenfelder-Oancea].

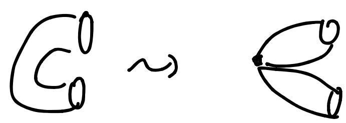
In fact, there is such an obstruction for Liouville manifolds in complete generality when $SH^\bullet(X) \neq 0$:

$\boxed{\text{"Degeneracy lemma"}}$ ("$SH^\bullet(X) = 0$ iff $RFH^\bullet(X) = 0$ for X Liouville")

Thm [Ritter '13]: For X Liouville, the image of $c^*: SH^*(X)^\vee[-2n] \to SH^\bullet(X)$ is nilpotent (w.r.t. pair-of-pants product).
Hence, c^* can only be an isomorphism if 1 is nilpotent $\iff SH^\bullet(X) = 0$.

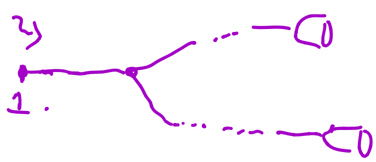
(Same property holds for the copairing on $HW^\bullet(L,L)$, w/ same proof, assuming L exact & each component non-compact).

Sketch: There is freedom to degenerate/pinch surfaces appearing in various counts, provided each component has ≥ 1 output:



$\Rightarrow$ c factors as: $\mathbb{K} \to H^\bullet(X) \otimes H^\bullet(X)[2n] \xrightarrow{\mathrm{accel}^{\otimes 2}} SH^\bullet(X) \otimes SH^\bullet(X)[2n]$

$1 \mapsto \Delta_!(1)$.



$\Rightarrow$ c^* factors as:

$$SH^\bullet(X)^\vee[-2n] \xrightarrow{\mathrm{accel}^\vee} H^\bullet(X)^\vee[-2n] \xrightarrow{\Delta_!(1)^*} H^\bullet(X) \xrightarrow{\mathrm{accel}} SH^\bullet(X)$$

P.D. $\wr\wr$ $H^\bullet_c(X)$ $\xrightarrow{\text{canon}}$ $H^\bullet(X)$

Rmk: this could break if X not exact! accel is then alg. map for quantum product.

(b) this map is an algebra map (for $\cdot$ & $\ast$), hence $\boxed{\text{preserves nilpotence condition}}$.

(a) this map is 0 in coh. degree 0 b/c every component of X non-compact, so lands in nilpotent elements (for cup product).

$\Rightarrow$ c^* has nilpotent image. $\boxtimes$

Now, pretend $SH^\bullet(X)$ finite dimensional $\overset{(\ast)}{\Rightarrow}$ it has a pairing compatible w/ copairing. Then, we would learn:

$SH^\bullet(X)$ fin. dim'l $\overset{(\ast)}{\Rightarrow}$ it has a pairing $\Rightarrow$ c_{SH} is non-degenerate $\Rightarrow$ $SH^\bullet(X) = 0$. (i.e., $SH^\bullet = 0$ or ∞!)

snake relation — by degeneracy lemma.

Similarly for $HW^\bullet(L,L)$.

There is no good reason I know that $(\ast)$ holds. However, it turns out that such an argument just works, provided stronger (categorical) finiteness hypotheses, which is the main observation.

Broadly the proof of Thm. (1) + (2) goes as follows:

Suppose $\mathcal{W}(X)$ (or $L \in \mathcal{W}(X)$) is proper, and X is non-degenerate. Then:

(1) (2)

(⋆) $\exists$ a pairing on $SH^\bullet(X)$ (resp. $HW^\bullet(L,L)$) fitting into the snake relation with the existing geometric copairing (degenerate!) c

$\Longrightarrow$ The copairing is perfect. (snake relation) why? more details below $\mathcal{G}$ or $\mathcal{C}$

$\Longrightarrow$ $SH^\bullet(X)$ (resp. $HW^\bullet(L,L)$) $=0$ ← (degeneracy Lemma – requires X exact Liouville, resp. L exact w/ all components non-compact.

$\Rightarrow$ $\mathcal{W}(X)$ (resp. L) $= 0$. $\boxed{2}$

(1)' & (2)': Translate the degeneracy Lemma into an alg. property of $\mathcal{W}(X)/L$ which localizes well/is inherited by summands, & implies "non-proper or zero".

<u>More about (1)</u>: We transfer the existence of pairing question to Hochschild homology as follows: ("has a hol. volume form" inducing 'cap' duality in Hochschild theory).

Under the given hypotheses, $\mathcal{OC}: HH_{*-n}(\mathcal{W}(X)) \xrightarrow{\simeq} SH^\bullet(X)$ and $\mathcal{W}(X)$ is "smooth Calabi-Yau". [G.]

- There is a purely algebraic construction of a copairing $c_{alg} \in HH_\bullet(\mathcal{C}) \otimes HH_\bullet(\mathcal{C})$ of any smooth category, [Shklyarov]:

$\mathcal{C}$ is smooth if its diagonal bimodule $\mathcal{C}_\Delta$ is perfect, i.e., split-gen. by Yoneda bimodules $\hom(-,L) \otimes \hom(K,-)$'s.

Any object $X \in \mathcal{D}$ induces $\mathbb{K} \xrightarrow{X} \mathcal{D}$ hence an element $ch_X :=$ image of 1 in $\mathbb{K} \cong HH_\bullet(\mathbb{K}) \xrightarrow{X_*} HH_\bullet(\mathcal{D})$

Smoothness therefore gives $ch_{\mathcal{C}_\Delta} \in HH_*(\mathrm{perf}(\mathcal{W}\text{-}\mathcal{W})) \underset{\text{Morita}}{\cong} HH_\bullet(\mathcal{W}^{op} \otimes \mathcal{W}) \cong HH_\bullet(\mathcal{W}) \otimes HH_\bullet(\mathcal{W})$.

- [Rezchikov, in prep] shows $\mathcal{OC}$ sends c_{alg} to c_{SH} under the given hypotheses.

- Now [Shklyarov] also shows that if $\mathcal{C} = \mathcal{W}(X)$ was proper, $\exists$ a pairing $HH_\bullet(\mathcal{C}) \otimes HH_\bullet(\mathcal{C}) \longrightarrow \mathbb{K}$ fitting into the snake relation w/ copairing c_{alg} (hence c_{SH}):

(induced by $\mathcal{W}^{op} \times \mathcal{W} \xrightarrow{\hom} \mathrm{perf}\,\mathbb{K} \subset \mathrm{Mod}\,\mathbb{K}$.

$\Rightarrow$ $\hom_*: HH_\bullet(\mathcal{W}^{op}) \otimes HH_\bullet(\mathcal{W}) \to HH_\bullet(\mathrm{perf}\,\mathbb{K}) \cong HH_\bullet(\mathbb{K}) \cong \mathbb{K}$).

$\cong$ $HH_\bullet(\mathcal{W}) \otimes HH_\bullet(\mathcal{W}) \xrightarrow{P}$ ... in contrast, $HH_\bullet(\mathrm{Mod}\,\mathbb{K}) = 0$

$\Rightarrow$ Thm part (1).

- Using the CY structure, which induces an iso. $HH_{\bullet-n}(\mathcal{W}(X)) \underset{CY}{\cong} HH^\bullet(\mathcal{W}(X))$, we can traslate the degeneracy Lemma property into a purely categorical statement, that has copairing ring w/ 1.

$$HH_*(\mathcal{W}(X))^\vee \xrightarrow{c_{alg}^*} HH_*(\mathcal{W}(X)) \underset{\cong}{\xrightarrow{CY}} HH^*(\mathcal{W}(X)) \text{ has nilpotent image.}$$

(Thm (1)').

This implies $\mathcal{W}(X)$ is non-proper or zero, and also persists under quotients ($\Rightarrow$ any quotient non-proper or zero).

(e.g., if $\mathcal{C}$ smooth, and $\mathcal{C} \to \mathcal{D}$ quotient then $f_*: HH_*(\mathcal{C}) \to HH_\bullet(\mathcal{D})$ sends $c_{alg}^{\mathcal{C}}$ to $c_{alg}^{\mathcal{D}}$).

<u>More about (2)</u>:

We'll appeal to the general fact [Brav-Dyckerhoff] that if $\mathcal{C}$ is a "smooth CY category" then any collection $\mathcal{P} \subseteq \mathcal{C}$ of right (resp. left) proper objects inherits a "proper CY structure," in particular there is a pairing $H^\bullet hom(K,L) \otimes H^\bullet hom(L,K) \longrightarrow \mathbb{K}[-n]$ for any $K, L \in \mathcal{P}$. (we just need $K = L$)..

$\dim(X) = 2n$

"integration pairing form of Poincaré duality" on Hochschild invts.

How to see this?

Say $L \in \mathcal{C}$ is right proper. Then denoting $\mathcal{P} = \{L\}$, hom gives

$$\mathcal{C}^{op} \times \mathcal{P} \xrightarrow{hom} perf\, \mathbb{K} \subset Mod\, \mathbb{K},$$ hence as before a pairing $HH_\bullet(\mathcal{C}) \otimes HH_\bullet(\mathcal{P}) \xrightarrow{\langle -,- \rangle} \mathbb{K}$

Now a smooth CY structure (more precisely, its "weak smooth CY structure" shadow) on a smooth category is $\Omega_{CY} \in HH_{*-n}(\mathcal{C})$ inducing (by cap product) $HH^*(\mathcal{C}, \mathcal{B}) \xrightarrow{\cap \Omega} HH_{*-n}(\mathcal{C}, \mathcal{B})$ for any $\mathcal{B}$.

(For $\mathcal{W}(X)$, [G.] shows $\Omega_{OC} := OC^{-1}(1)$ is such a (weak) smooth CY structure).

Get the desired pairing p on L (if L proper) by

$$p_{L,L}: H^\bullet(hom(L,L)) \otimes H^\bullet hom(L,L) \xrightarrow{[\mu^2]} H^\bullet(hom(L,L)) \xrightarrow{[incl.]} HH_\bullet(\mathcal{P}) \xrightarrow{\langle \Omega, - \rangle} \mathbb{K}.$$

<u>To check</u>:

(a) (algebraic fact): Any (weak) smooth CY $(\mathcal{C}, \Omega)$ inherits (algebraically from the element Ω) a copairing $\forall K,L$: $c^{\Omega}_{K,L} \in H^* hom(K,L) \otimes H^* hom(L,K)\,[n]$. If $L \in \mathcal{C}$ is a proper object, then $c_{L,L}$ satisfies snake relation with $p_{L,L}$ (also constructed using Ω).

(b) (<u>Geometric</u> fact): On $(\mathcal{W}(X), \Omega_{OC})$, the geometric copairing

$c^{HW^\bullet}_{K,L}$ defined by counting $\subset$ coincides with the algebraic ($c^{\Omega_{OC}}_{K,L}$) copairing.

(implicit in [G.], which describes an inverse to $- \cap \Omega_{OC}$ in terms of counting disks w/ two outputs)

(a)+(b) $\Rightarrow$ if $L \in \mathcal{W}(X)$ a proper object, $\exists$ a pairing on $HW^\bullet(L,L)$ which is compatible w/ (satisfies snake relation w/) the geometric copairing, as needed. $\Rightarrow$ Thm (2).

(Moreover, the degeneracy lemma by (b) implies $c^{\Omega_{OC}}_{L,L}$ is degenerate, a property which is inherited by summands & implies such summands are non-proper or zero).
$\Rightarrow$ Thm (2').

(a)+(b): How to define $c^{\Omega}_{K,L}$? $(-\cap\Omega)$ induces $HH^\bullet(\mathcal{C}, \underbrace{hom(-,L)\otimes hom(L,-)}_{\mathcal{Y}_{L,L}}) \xrightarrow{\cong} H^\bullet hom(L,L)$

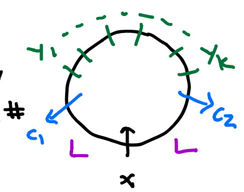
Taking $(-\cap\Omega)^{-1}([id_L])$ gives an element $\tilde{c}_L \in HH^n(\mathcal{C}, hom(-,L)\otimes hom(L,-))$, whose 'leading order piece' (using $HH^\bullet(\mathcal{C},\mathcal{B}) \to H^\bullet\mathcal{B}(K,K)$) is $\tilde{c}_{K,L} \in H^\bullet hom(K,L)\otimes H^\bullet hom(L,K)[n]$

Call this $c^{\Omega}_{K,L}$.

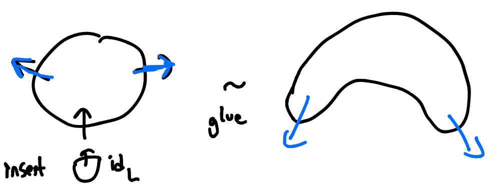
In [G.] I proved that an inverse to $-\cap\Omega_{oe}$: $HW^\bullet(L,L) \longrightarrow HH^\bullet(\mathcal{W}(X), CW^\bullet(-,L)\otimes CW^\bullet(L,-))$

can be geometrically defined by counting:

$$\underset{hom(L,L)}{\underset{\cup}{x}} \longmapsto \Big\{ \underset{hom(X_0,X_1)\otimes\cdots\otimes hom(X_{k-1},X_k)}{\underset{\cup}{(y_1,\dots,y_k)}} \longmapsto \underset{hom(X_0,L)\otimes hom(L,X_k)}{\underset{\cup}{\sum_{c_1,c_2}\Big(\#\ \text{[disk]}\Big)\, c_1\otimes c_2}} \Big\}$$



Plugging in $x = id_L$ (a representative), & looking at length $k=0$ piece gives precisely the moduli spaces



appearing in copairing.

$$\boxed{\text{Thank you!}}$$

Discussion: What about the case of stops / sectors? In general there can exist non-compact Lagrangians which are proper non-zero objects, so precise argument above doesn't work (e.g., thimbles in a Fukaya-Seidel category).

Do $\mathcal{W}(Y, \text{stops})$ ever have CY structures? Sometimes, for example:

X^{2n} Weinstein m'fold $\rightsquigarrow$ $(X\times(\mathbb{C}, 2pt))^{2n+2}$. know $\mathcal{W}(X)\cong\mathcal{W}(X\times(\mathbb{C},2pts))$

$\Rightarrow$ $\mathcal{W}(X\times(\mathbb{C},2pts))$ has a CY structure of dimension n, inherited from $\mathcal{W}(X)$;

but one needs to see this from the '<u>relative</u> CY <u>structure</u>' on this cat. (rel. stop).

Have $\underset{\text{boundary cat.}}{\mathcal{A}_\partial} \xrightarrow{i} \underset{\text{'cat. of interior'}}{\mathcal{A}_{int}}$, & the Poincaré duality rel boundary, consider the complex cone $(HH_\bullet(\mathcal{A}_\partial) \xrightarrow{i_*} HH_\bullet(\mathcal{A}_{int}))$

($HH_\bullet$ chain complex)

want Ω to live here for a rel. smooth CY str. (the 'weak' version).