Categorical non-popperness in wrapped Floer theory

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Goal: discuss non-fractured (non-popperness) phenomenon frequently seen in wrapped Floer homology, and show that to a large part they are a necessary consequence of global categorical TFT structures.

The phenomenon

\((X^2, \lambda)\) Liouville manifold \(\quad (\omega = d\lambda \text{ symplectic, st. framing out by } \mathbb{Z} \text{ present } X \text{ off sheet as symplectization of orbit manifold, } \Xi X).\)

Assume \(n > 0\) (so \(X\) is non-compact).

\[\rightsquigarrow\] Symplectic cohomology \(SH^i(X)\), can arrange \(SH^i(X) = H^i(C^*(X) \oplus C_0(S^1) \oplus \text{Reeb orbits})\)

\[\rightsquigarrow\] wrapped Fukaya category \(\mathcal{W}(X)\)

- objects: \(X \in X\), exact Lagrangian, properly embedded (possibly non-compact but)
  - cylinders \@ \infty,
- cohomological spheres: \(HW^*(L_0, L_\infty) := \lim_{\rightarrow \infty} HF^*(L_0, L_\infty),\)

- can arrange \(HW^*(k, L) = H^i(C^*(k, L) \oplus K_0 \text{ Reeb chords } 2\omega \otimes 2\omega_{L, L}^3, \delta)\)

\[C^0(L) \text{ if } k = L\]

The presence of chords/orbits in these complexes means that \(SH^*/HW^*\) could be infinite dimensional but doesn't force infinite dimensionality.

\[\text{Ex.}: X = T^2 S^1\]

\[\text{Reeb (pts.)}\]

\[\text{Hw}^*(L) = \lim_{\rightarrow \infty} HF^*(L^0, L) = \mathcal{C}(S^2) \text{ (} = \text{Ext}_{C_*}([0^*, 0^*], [0^*, 0^*]) \text{)}\]
On the other hand, in all known explicit computations on Liouville manifolds,
\[ \text{(to my knowledge)} \]

(a) \( \text{SH}^1(X) = 0 \) or is infinite rank.

(b) for a non-compact connected \( L, \text{HW}^*(L, L) = 0 \) or \( \infty \).

\[ \text{Note if } K \text{ or } L \text{eps} \text{, or more generally } \exists \text{K and } \exists \text{L lie on different components of } \exists \text{X,}
\]
then \( \text{HW}^*(K, L) = 0 \).

\[ \text{Ex: } \]
\[ \text{HW}(K, L) = 0. \]

\[ \text{Note: } \text{HW}^*(K, L) \text{is a nevertheless `generic' infinite.} \]

**Folk Q: Must these `infinite or zero' phenomena hold?**

**Ranks:**
- (c) is the special case \( L = \Delta \times X = X \) of (b).
- (d) is not special case \( L = \Delta \times X \) of (b).
- (e) is special case \( L = \Delta \times X \) of (b).

\[ \text{unknown even for every } X = T^*Q \text{ that } \text{SH}^1(X) \cong \text{HW}^*(L, L) \text{ is always infinite, to my}
\]
understanding. (Known if \( \pi(Q) = 0 \) or \( Q \text{ nilpotent,}
\]
also if \( L \text{ has only many co-dense classes} \).

- (f) If (a) held, then Viterbo's accelerator map \( H^*(X) \to \text{SH}^1(X) \) cannot be an iso. (Viterbo(Davis-Moore))

\[ \Rightarrow \exists \text{ a Rank orbit on } \exists \text{X (Weinstein cont).} \]

- On the other hand, [Ritter, Ritter-Smith]: on some negative line bundles, a class of monotone
- non-exact non-compact (contact at oo) sympl. manifolds, \( \text{SH}^1(X) = \text{finite non-zero} 
\]
- \( \Rightarrow \exists \text{ a Rank orbit on } \exists \text{X.} \)

It seems unclear whether these hold. However, our main result is that a more `global' version
of the `infinite or zero' phenomena holds in broad generality (in the exact setting):

\[ \text{means } \exists \text{ a collection of Lagrangians satisfying Aubin's genericity criterion (OE: } \text{HW}^*(X) \to \text{SH}^1(X) \text{ hits L) } \]

\[ \text{Weinstein } \Rightarrow \text{non-degenerate.} \]

**Then (** Gil, in prep **)**: Let \( X \) be a non-degenerate Liouville manifold, e.g., any Weinstein manifold of dim \( > 1 \).

Then:
- (1) \( \text{W}^*(X) \) is either \( 0 \) or non-paper. \( \Rightarrow \text{if } \pi(X) \neq 0 \exists K, L \text{ with } \text{HW}^*(K, L) = 0, \text{including a pair in any (split)-generating set.} \)

- (2) Any \( L \in \text{W}^*(X) \) of whose components are non-compact is either \( 0 \) or both, left or right non-paper.

\[ \Rightarrow \text{if } L \neq 0, \exists K, K' \text{ with } \text{HW}^*(K, K') = 0, \text{including one in any (split)-generating set.} \]

- (2') Same for any summand \( K \neq L \) in perf \( W^*(X) \).

\[ \text{Ranks: Say } \pi(X) \text{ gen. by } \{ \Delta_i \}_{i=1}^k \text{ with each } \Delta_i \text{ non-compact (e.g., cocores of Weinstein pers. [CDS, p91]).}
\]

- Assume we've discounted all \( \Delta_i \text{ which are zero). Then if } \pi(X) \neq 0 \Rightarrow \exists \text{ some } \text{HW}^*(K, L) = 0. \]
The proofs appeal to the structure and properties of TQFT operators, and their interplay with the fact that $\mathcal{H}(X)$ is a smooth Calabi-Yau category \cite{G.}.

TQFT gadgets: Recall \cite{Rita} that $\mathcal{H}(X)$ resp. $\mathcal{H}^*(X)$ admit (closed resp. open)

2d TQFT operators, as is common in Floer theory; e.g., a picture in genus $0$.

E.g., on $\mathcal{H}^*(X)$:

\[ \begin{array}{c}
\text{inputs} \\
\tau \\
\text{n inputs} \\
\end{array} \qquad \begin{array}{c}
\text{outputs} \\
\tau' \\
\text{n outputs} \\
\end{array} \]

\[ \begin{array}{c}
\text{resp.} \\
\text{L} \\
\text{product} \\
\end{array} \qquad \begin{array}{c}
\text{SH}^*(X)^{\otimes_1} \\
\text{resp.} \\
\text{HW}^*(X)^{\otimes_1} \\
\end{array} \]

\[ \begin{array}{c}
\text{compatibility with gluing surfaces} \\
\end{array} \]

and

\[ \begin{array}{c}
\text{E.g., on } \mathcal{H}^*(X): \\
0 \otimes 1 \leftrightarrow \text{id} \\
0 \otimes 0 \leftrightarrow \text{unit} \\
0 \otimes 1 \leftrightarrow \text{product} \\
0 \otimes 0 \leftrightarrow \text{pairing} \\
\end{array} \]

\[ \begin{array}{c}
\text{compatibility with gluing surfaces} \\
\end{array} \]

\[ \begin{array}{c}
\text{There's an important restriction: } n_2 = \# \text{ adps } \geq 1. \text{ A technical problem: (failure of maximum principle)} \end{array} \]

In setting up a pairing, reflecting a more fundamental point: if $\exists$ a compatible pairing

\[ p = \langle -, \cdot \rangle : (\mathcal{H}^*)^{\otimes_2} \to K, \text{ then the snake relation in TQFT } \Rightarrow \]

\[ \begin{array}{c}
\text{id} \\
\leftarrow \\
\text{p} \\
\end{array} \]

\[ v = \text{id}(v) = (\text{ pair } \cdot \text{id }) (v \circ c) = \sum \langle v, a_i \rangle b_i. \]

\[ \Rightarrow \mathcal{H}^*(X) \text{ finite-dimensional and the maps } \mathcal{H}^*(X) \to \mathcal{H}^*(X) \]

are inverse $\Rightarrow c$ is non-degenerate.
So, having a pairing implies the copairing c is non-degenerate, & finite-dimensionality.

Ideas: The failure of c to be non-degenerate (c* to be an isom.) obstructs the existence of a pairing.

\[ \text{Cone}(c^*; SC^*(X)[2n]) \rightarrow SC^*(X) \text{ non-zero.} \]

\[ \text{identifying } SC^*(X)^[-2n] 
\]

In fact, there is such an obstruction for Liouville manifolds in complete generality when SH*(X) \(\neq 0\).

**Degeneracy Lemma**

\[ "SH^*(X) = 0 \iff RFH^*(X) = 0 \text{ for } X \text{ Liouville"} \]

Thus [Ritter '03]: For X Liouville, the image of c*: SH*(X)[2n] \(\rightarrow\) SH*(X) is nilpotent w.r.t. \(\otimes\).

Hence, c* can only be an isomorphism if 1 is nilpotent. \(\Rightarrow\) SH*(X) = 0.

(Same property holds for the copairing on HW*(L,L), w/ same proof, assuming L exact & each component non-compact).

Sketch: There is freedom to degenerate/pinch surfaces appearing in various counts, provided each component has \(\geq 1\) output:

\[ \begin{array}{c}
\text{\(c^0 \sim \otimes\)} \\
\Rightarrow \text{c factors as: } K \rightarrow H^*(X) \otimes H^*(X)[2n] \rightarrow \text{SH*(X) } \otimes \text{SH*(X)}[2n] \\
\Rightarrow \text{c* factors as: } \\
\end{array} \]

\[ \text{SH*(X)[2n] } \rightarrow H^*(X)[2n] \rightarrow \text{SH*(X)} \rightarrow \text{SH*(X)} \text{ nilpotent.} \]

(\(b\) this map is an algebra map for \(\otimes\),

takes nilpotent to nilpotent).

\[ \text{H*(X)} \]

\[ \Rightarrow \text{c has nilpotent image.} \]

Now, pretend SH*(X) finite dimensional \(\Rightarrow\) it has a pairing compatible c/copairing. Then, we would learn:

\[ \text{SH*(X) fin. dim. } \Rightarrow \text{it has a pairing } \Rightarrow \text{c* is non-degenerate } \Rightarrow \text{SH*(X) = 0. (i.e., SH* = 0 or 2n.)} \]

Similarly for HW*(L,L).

There is no good reason I know that \((b)\) holds. However, it turns out that such an argument just works, provided stronger (categorical) finiteness hypotheses, which is the main observation.

Broadly the proof of Thm. (1) + (2) goes as follows:

Suppose W(X) (or L \(\in\) W(X)) is proper, and X is non-degenerate. Then:

\[ \begin{array}{c}
(1) \\
(2) \\
\end{array} \]
The pairing is perfect. (Snake relation) why? more details below $\mathcal{B}$ or $\mathcal{C}$

$\Rightarrow$ SH$^*$ (resp. HW$^*$) = 0 (degeneracy lemma - requires X exact L-wittle, resp. L exact w/ all components non-compact).

$\Rightarrow$ $W(X)$ (resp. $L$) = 0.

1) & 2): Translate the degeneracy lemma into a diag. property of $W(X)/L$ which localizes well/is inherited by summands, & implies “non-proper or zero”.

More about 1): We transfer the existence of pairing question to Hochschild homology as follows:

Under the given hypotheses, $\mathcal{D}$: $HH_{n-m}(W(X)) \rightarrow SH^*(X)$ and $W(X)$ is “smooth 1-shifted” $\mathcal{C}$.

- There is a purely algebraic construction of a pairing $c_{\varphi}(\mathcal{E}) \in HH_*(\mathcal{E}) \otimes HH_*(\mathcal{E})$ for any smooth category $\mathcal{E}$.

$\mathcal{E}$ is smooth if its diagonal bimodule $\mathcal{E}_0$ is perfect, i.e., split-gen. by Yoneda bimodules $\text{hom}(-, L) \otimes \text{hom}(L, -)$.

Any object $x \in \mathcal{E}$ induces $K \xrightarrow{x} \mathcal{E}$ hence an element $\text{ch}_x := \text{image of } 1 \in K \simeq HH_*(K) \xrightarrow{x} HH_*(\mathcal{E})$.

Smoothness therefore gives $\text{ch}_x \simeq HH_*(\text{perf}(W \otimes W)) \cong HH_*(W \otimes W) \cong HH_*(W)$.

- [Rezchikov 1/prep] shows OE sends $c_{\varphi}$ to $c_{W}$ under the given hypotheses.

Now [Shklyaev] also shows that if $\mathcal{E} = W(X)$ was proper, $\exists$ a pairing $HH_*(\mathcal{E}) \otimes HH_*(\mathcal{E}) \rightarrow K$ fitting into the snake relation w/ copairing $c_{\varphi}$ (hence $c_{W}$):

(Thm part 1).

- Using the $\mathcal{C}$Y structure, which induces an iso. $HH_*(W(X)) \cong HH_*(W(X))$, we can translate the degeneracy lemma property into a purely categorical statement, that $\mathcal{E}$ has copairing $c_{\varphi}$ w/ L.

This implies $W(X)$ is non-proper or zero, and also persists under quotients (e.g., quotient of $\mathcal{E}$ is proper).

More about 2):
We'll appeal to the general fact [Brau-Deckerhoff] that if \( C \) is a smooth CY category then any collection \( \mathcal{P} \subseteq C \) of right (resp. left) proper objects inherits a "proper CY structure," in particular there is a pairing \( H^*\text{hom}(k_L) \otimes H^*\text{hom}(L,k) \to \text{Ik}[-n] \) for any \( k,L \in \mathcal{P} \). (we just need \( K = L \)).

How to see this?

Say \( L \in \mathcal{P} \) is right proper. Then denoting \( \mathcal{P} = \{ L \} \), hom gives

\[
\mathcal{P}^\text{op} \times \mathcal{P} \xrightarrow{\text{hom}} \text{perf} K \subset \text{Mod} K, \quad \text{hence as before a pairing } H^i(E) \otimes H^j(E) \to \text{Ik}
\]

Now a smooth CY structure (more precisely its "weak smooth CY structure" shadow) on a smooth category is \( \Omega_{CY} \in H^{i-n}(C) \) inducing (by coproduct) \( H^i(C,B) \to H^i_{cy}(C,B) \) for any \( B \).

(For \( W(X) \), [G.] shows \( \Omega_{\text{log}} := 0 \Omega^{\text{-3}}(1) \) is such a (weak) smooth CY structure.)

Get the desired pairing \( p \) on \( L \) (if \( L \) proper) by

\[
P_L : H^i(\text{hom}(L,L)) \otimes H^j(\text{hom}(L,L)) \xrightarrow{[\alpha, \beta]} H^i(\text{hom}(L,L)) \xrightarrow{[\iota\otimes 1]} H^i(\text{B}) \xrightarrow{1 \otimes \eta} \text{Ik}
\]

To check:

(a) (algebraic fact): Any (weak) smooth CY \( (C, \Omega) \) inherits (algebraically from the element \( \Omega \)) a pairing \( \forall k,L : c_{k, L} \in H^i(\text{hom}(k, L)) \otimes H^j(\text{hom}(L, k)) [n] \). If \( L \in \mathcal{P} \) is a proper object, then \( c_{k, L} \) satisfies snake relation with \( p_{k, L} \) (also constructed using \( \Omega \)).

(b) (geometric fact): On \( (W(X), \Omega_{\text{log}}) \), the geometric copairing \( c_{kL}^{\text{log}} \) defined by counting \( C \) coincides with the algebraic \( c_{kL}^{\text{log}} \) copairing. (implicit in [G.], which describes an inverse to \( \sim \Omega_{\text{log}} \) in terms of counting disks \( v \) to two outputs)

(a) + (b) \( \Rightarrow \) if \( L \in W(X) \) a proper object, \( \exists \) a pairing on \( H^i(\text{hom}(L, L)) \) which is compatible w/ (satisfies snake relation w/) the geometric copairing, as needed \( \Rightarrow \text{Thm (2)}. \)

(Moreover, the degeneracy lemma by (6) implies \( c_{L, L}^{\text{log}} \) is degenerate, a property which is inherited by summands \( & \) implies such summands are non-paper or \( \Omega \).

\( \Rightarrow \text{Thm (2')}\).
How to define $C_{k,L}^{\mathbf{g}}$? $(-\otimes \Omega)$ induces $\text{HH}^n(\mathcal{C}, \text{hom}(\mathcal{L}, L) \otimes \text{hom}(L, \mathcal{L})) \rightarrow \text{HH}(\mathcal{L}, L)$.

Taking $(-\otimes \Omega)^{-1}([\text{id}_L])$ gives an element $\zeta \in \text{HH}^n(\mathcal{C}, \text{hom}(\mathcal{L}, L) \otimes \text{hom}(L, \mathcal{L}))$, whose `leading order piece' (using $\text{HH}(\mathcal{C}, \Omega) \rightarrow \Omega_0(\mathcal{L}, \mathcal{L})$) is $C_{k,L} \in \text{H}^n(\mathcal{L}, L) \otimes \mathbb{H}(\mathcal{C}, \mathcal{L})$.

Call this $C_{k,L}^{\mathbf{g}}$.

In [6.] I proved that an inverse to $-\otimes \Omega_0^*: \mathbb{H}^*(\mathcal{L}, L) \rightarrow \text{HH}^*(\mathcal{C}, \Omega(\mathcal{L}, L) \otimes \mathbb{H}^*(\mathcal{C}, \mathcal{L}))$

can be geometrically defined by counting:

$\overset{x = \text{id}_L}{\sim} \overset{\mathcal{C}_1 \otimes \mathcal{C}_2}{\text{hom}(\mathcal{L}, \mathcal{L})} \overset{\text{hom}(\mathcal{L}, \mathcal{C}) \otimes - \otimes \text{hom}(\mathcal{C}, \mathcal{L})}{\rightarrow} \sum_{\mathcal{C}_1 \otimes \mathcal{C}_2} \left( \begin{array}{c}
\text{hom}(\mathcal{C}_1, \mathcal{C}_2) \otimes \mathbb{H}^*(\mathcal{C}, \mathcal{L})
\end{array} \right)$

Plugging in $x = \text{id}_L$ (a representation), & looking at length, $k=0$ piece gives precisely the moduli spaces appearing in repairing.

$\overset{\text{X}^2, \text{Witten}}{\sim} \overset{\text{X}^2, \text{Scaly}}{\text{Witten}}$

**Discussion**: What about the case of steps/sectors? In general there can exist non-compact Lagrangians which are proper non-zero objects, so precise argument above doesn't work (e.g., thimbles in a Fukaya-Seidel category).

Do $W(\text{steps})$ ever have CY structures? Sometimes, for example:

$\overset{\text{X}^2, \text{Witten}}{\sim} \overset{\text{X} \times (\mathbb{C}, 2\mathbb{P}^1)}{\text{X}^2 \text{Witten m'fold}} \overset{\text{X} \times (\mathbb{C}, 2\mathbb{P}^1)}{\rightarrow} (\mathbb{C}, 2\mathbb{P}^1) \rightarrow \text{X} \times (\mathbb{C}, 2\mathbb{P}^1)$

$\overset{\text{W}^*(\text{X}) \equiv \text{W}^*(\text{X} \times (\mathbb{C}, 2\mathbb{P}^1))}{\text{W}^*(\text{X}) \equiv \text{W}^*(\text{X} \times (\mathbb{C}, 2\mathbb{P}^1))}$

$\overset{\text{Y} \rightarrow \text{W}(\text{X} \times (\mathbb{C}, 2\mathbb{P}^1)) \text{ has a CY structure of dimension n, inherited from W(X),}}{\text{Y} \rightarrow \text{W}(\text{X} \times (\mathbb{C}, 2\mathbb{P}^1)) \text{ has a CY structure of dimension n, inherited from W(X),}}$

but one needs to see this from the 'relative CY structure' on this cat. (rel. step).

Have $A_2 \rightarrow \text{mon}, \quad \overset{\text{boundary}}{\Rightarrow} \text{Y} \rightarrow \text{Poncare duality rel boundary, consider the complex}$

boundary cat. 

"Out of mind"
cone \( \cd (\text{HH}_2(A_\Omega) \rightarrow \text{HH}_0(\text{A}_{\text{int}})) \)

\( (\text{with chain complex}) \)

\( \uparrow \text{want } \Omega \text{ to live here for a rel. smooth CY str. (the `weak' version) } \)