# Relative Quantum Cohomology

*and other stories*

Joint work with Jake Solomon

$(X,\omega)$ closed, symp., $J$ a.c.s.

**Closed GW:**

Nov. ring

$$*: H^*(X;Q)^{\otimes 2} \longrightarrow H^*(X;Q)$$

assoc. $\Longleftrightarrow$ WDVV

PDE in $\Phi = \sum \cdots GW(\cdots)$

**Open GW:** $L \subset X$ Lag.

$$\wedge: QH^*_u(X,L)^{\otimes 2} \longrightarrow QH^*_u(X,L)$$

assoc. $\Longleftrightarrow$ OWDVV

PDE in $\Phi = \sum \cdots GW(\cdots)$, $\overline{\Omega} = \sum \cdots OGW(\cdots)$

$\Uparrow$ $\quad \Nearrow$

$\boxed{\text{commutativity of partial derivatives of נ} \in \mathrm{End}(\text{cone})}$

**Plan:**
- Define the cone
- Define נ
- Formulate key properties of נ
- Formulate OWDVV
- Define מ
- Sample computation of $QH^*(X,L)$

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**Setting:** $(X,\omega)$ closed, symplectic, $L \subset X$ rel-spin Lag., $J$ $\omega$-tame a.c.s.

$$T^{\beta_1} \cdot T^{\beta_2} = T^{\beta_1+\beta_2}$$

**Rings:**

$$\Lambda = \left\{ \sum_{i=0}^{\infty} a_i T^{\beta_i} \;\middle|\; \begin{array}{ll} a_i \in \mathbb{R} & \omega(\beta_i) \geq 0 \\ \beta_i \in H_2(X,L;\mathbb{Z}), & \lim_{i\to\infty} \omega(\beta_i) = \infty \end{array} \right\}$$

$$\pi: H_2(X;\mathbb{Z}) \longrightarrow H_2(X,L;\mathbb{Z})$$

$\mathbin{\rotatebox[origin=c]{90}{$\subseteq$}}$

$$\Lambda_c = \left\{ \sum_{i=0}^{\infty} a_i T^{\pi(\beta_i)} \;\middle|\; \begin{array}{ll} a_i \in \mathbb{R} & \omega(\beta_i) \geq 0 \\ \beta_i \in H_2(X;\mathbb{Z}), & \lim_{i\to\infty} \omega(\beta_i) = \infty \end{array} \right\}$$

$$\deg T^{\beta} := \mu(\beta)$$

$\hat{H}^*(X,L;\mathbb{R})$

data about $L$.

$V$, $S$ graded vector spaces ... bases

$\hat{H}^*(X,L;\mathbb{R})$ about $L$.

$W, S$, graded vector spaces$/\mathbb{R}$ $\quad w_1, \ldots, w_N \in W \quad v_1, \ldots, v_m \in S$ bases

$$\rightsquigarrow \quad R_W := \Lambda \otimes \mathbb{R}[[W[2] \oplus S[1]]] \simeq \Lambda[[t_1, \ldots, t_N, s_1, \ldots, s_m]]$$
$$\cup\!\!\!\!|$$
$$Q_W := \Lambda_c \otimes \mathbb{R}[[W[2]]] \simeq \Lambda_c[[t_1, \ldots, t_N]]$$

Cone:

$$\underline{i} : A^*(X; Q_W) \longrightarrow R_W[-n] \qquad i : L \hookrightarrow X$$
$$\eta \longmapsto \int_L i^*\eta$$

$$\mathrm{Cone}(\underline{i}) = \underbrace{A^*(X; Q_W)}_{\text{GW thy lives here}} \oplus \underbrace{R_W[-n-1]}_{\text{input into OGW}}, \quad d_c(\eta, \xi) = (d\eta, \underline{i}(\eta))$$

Remark: $i : L \hookrightarrow X \rightsquigarrow i^* : A^*(X;\mathbb{R}) \longrightarrow A^*(L;\mathbb{R})$

$$\underline{i}_{\mathbb{R}} : A^*(X;\mathbb{R}) \longrightarrow \mathbb{R}$$
$$\eta \longmapsto \int_L i^*\eta .$$

$\mathrm{Cone}(\underline{i}) = A^*(X) \oplus \mathbb{R}[-n-1]$

$d_{cone}(\eta, \xi) = (d\eta, \underline{i}(\eta) - \cancel{d\xi}) = (d\eta, \underline{i}(\eta))$

$H^*(\mathrm{Cone}(\underline{i})) = H^*(\hat{A}(X))$

$\hat{A}(X) = \{\eta \in A^*(X) \mid \int_L \eta|_L = 0\}$.

## The endomorphism $J : \mathrm{Cone}(\underline{i}) \circlearrowleft$ :

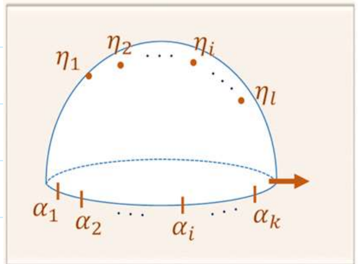
Step 1: sphere and disk operations.

$$q_{k,l} : A^*(L; R_W)^{\otimes k} \otimes A^*(X; Q_W)^{\otimes l} \longrightarrow A^*(L; R_W)$$

$\beta \in H_2(X, L; \mathbb{Z}) \rightsquigarrow \overline{\mathcal{M}}_{\ldots}(\beta) = \{u : (D, \partial D) \longrightarrow (X, L) \mid u \text{ J-hol.}, \deg u = \beta \ldots\}$, $\xrightarrow{evb_j} L \quad j = 0, \ldots, k$

$$\beta \in H_2(X,L;\mathbb{Z}) \rightsquigarrow \quad \overline{\mathcal{M}}_{k+1,l}(\beta) = \left\{ \begin{array}{l} u:(D,\partial D) \to (X,L) \\ z_0,\dots,z_k \in \partial D,\ w_1,\dots,w_l \in D \end{array} \middle| \begin{array}{l} u \text{ J-hol.} \\ \deg u = \beta \\ z_i \neq z_j,\ w_i \neq w_j \end{array} \right\} \Big/ \sim \quad \xrightarrow{evb_j} \quad \xrightarrow{evi_j} X \quad j=1,\dots,l$$

$$q_{k,l}(\alpha_1,\dots,\alpha_k;\eta_1,\dots,\eta_l) := \sum_{\beta \in H_2(X,L;\mathbb{Z})} T^{\beta} (evb_0)_* \left( \bigwedge_{j=1}^{k} evb_j^* \alpha_j \wedge \bigwedge_{j=1}^{l} evi_j^* \eta_j \right) + \delta_{k,1} \cdot \delta_{l,0} \cdot d\alpha_1 .$$



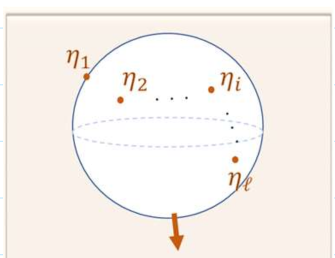

$$q_{-1,l}: A^*(X;Q_W)^{\otimes l} \longrightarrow R_W$$

$$q_{-1,l}(\eta_1,\dots,\eta_l) = \sum_{\beta} T^{\beta} \int_{\overline{\mathcal{M}}_{0,l}(\beta)} \bigwedge_{j=1}^{l} evi_j^* \eta_j$$

$$q_{\emptyset,l}: A^*(X;Q_W)^{\otimes l} \longrightarrow A^*(X;Q_W)$$

$$\beta \in H_2(X;\mathbb{Z}) \quad \overline{\mathcal{M}}_{l+1}(\beta) = \left\{ \begin{array}{l} u: S^2 \to X \\ w_0,\dots,w_l \in S^2 \end{array} \middle| \begin{array}{l} \deg u = \beta \\ u \text{ J-hol} \\ w_i \neq w_j \end{array} \right\} \Big/ \sim \quad \xrightarrow{ev_j} X \quad j=0,\dots,l.$$

$$q_{\emptyset,l}(\eta_1,\dots,\eta_l) = \sum_{\beta \in H_2(X;\mathbb{Z})} T^{\pi(\beta)} (ev_0)_* \left( \bigwedge_{j=1}^{l} ev_j^* \eta_j \right)$$




**Step 2:** Bounding pairs.

$$R_W = \Lambda[[t_1,\dots,t_N,s_1,\dots,s_M]] \quad , \quad Q_W = \Lambda_c[[t_1,\dots,t_N]]$$

Let $\qquad \Lambda^+ \lhd \Lambda, \quad \Lambda_c^+ \lhd \Lambda_c, \quad m_W \lhd \mathbb{R}[[t_1,\ldots,t_N]], \quad m_S \lhd \mathbb{R}[[s_1,\ldots,s_M]]$

be the maximal ideals.

Set

$$K_W = \Lambda^+ R_W + m_W R_W + m_S R_W \quad \lhd \quad R_W$$

$$I_W = \Lambda_c^+ Q_W + m_W Q_W \quad \lhd \quad Q_W \; .$$

Definition: $(\gamma, b) \in I_W A^*(X; Q_W) \oplus K_W A^*(L; R_W)$ is a bounding pair if

$d\gamma = 0, \quad |\gamma| = 2, \quad |b| = 1,$ and

$$\exists c \in K_W \text{ s.t. } |c| = 2 \text{ and } \sum_{k,l \geq 0} \frac{1}{l!} \mathfrak{q}_{k,l}(b^{\otimes k}; \gamma^{\otimes l}) = c \cdot \underset{A^*(L)}{\underset{\cap}{1}} \; .$$

Equiv.: $b$ is a weakly bounding cochain for $\mathfrak{m}^\gamma$,
$c$ is the MC const. (= the "open" potential)

Step 3: the endomorphism.

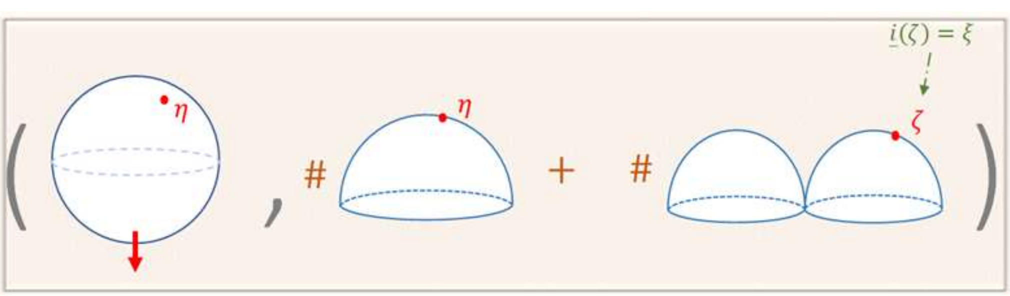
for a bd pair $(\gamma, b)$,

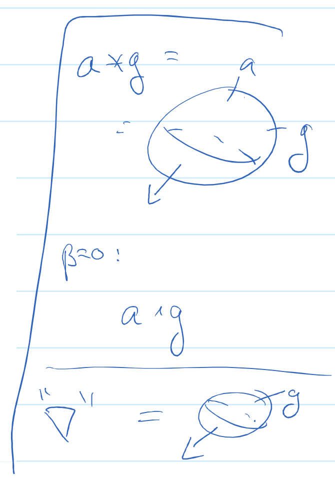
$$J = J^{b,\gamma} : \underset{A^*(X;Q) \oplus R[n-1]}{\underset{\shortparallel}{\mathrm{Cone}(\underline{i})}} \longrightarrow \mathrm{Cone}(\underline{i})$$

$\langle \alpha, \beta \rangle = \int_L \alpha \wedge \beta$

$$J(\eta, \xi) = \Big( \sum_{l \geq 0} \frac{1}{l!} \mathfrak{q}_{\phi, l+1}(\eta \otimes \gamma^{\otimes l}), \sum_{k,l \geq 0} \frac{1}{l!(k+1)} \langle \mathfrak{q}_{k,l+1}(b^{\otimes k}; \eta \otimes \gamma^{\otimes l}), b \rangle + \sum_{l \geq 0} \frac{1}{l!} \mathfrak{q}_{-1,l+1}(\eta \otimes \gamma^{\otimes l}) + c \cdot \xi \Big)$$

$$= \big( \mathfrak{q}^{\gamma}_{\phi,1}(\eta) \quad , \quad \mathfrak{q}^{\gamma}_{-1,1}(\eta) + c \cdot \xi \big)$$



$$c\cdot\xi = c\cdot\int_L i^*\xi = \langle c\cdot 1, i^*\xi\rangle = \langle c\cdot 1, \ \cdot\ \rangle$$

$$= \langle\ \cdot\ ,\ \cdot\ \rangle$$

"$\nabla$" $=$ $g$

$\Phi = \sum \cdots GW(\cdots)$

$\nabla\Phi =$

$\langle\nabla\Phi, \eta\rangle_X =$ $\eta$

**Theorem:** 1) $]$ is a chain map

2) $(\gamma,b)\sim(\gamma',b) \Longrightarrow ]^{b,\gamma} \sim ]^{b',\gamma'}$

gauge equiv. ... chain homotopic.

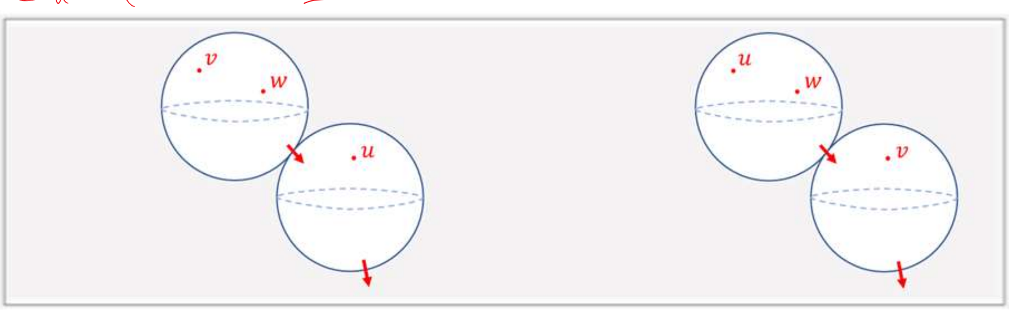
**Main theorem:** The induced map on coh. satisfies

$$\partial_u ] \circ \partial_v ] = \partial_v ] \circ \partial_u ] , \qquad \forall u,v \in (Q_W\otimes W)\oplus(R_W\otimes S).$$

**Sample Cor. — Meaning in the first component:**

$(\partial_u ])_1(\eta) = u$ $\eta$

$$\big[\partial_u ] (\partial_v ](w))\big]_1 = \big[(\partial_v ])(\partial_u ](w))\big]_1$$




$\Updownarrow$

assoc. of $*$ $\qquad u*(w*v) = (u*w)*v$

$\Uparrow$

$$\Updownarrow$$

WDVV.

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**Assumptions:** 1) $U \subset H^*(X;\mathbb{R})$ a linear subspace s.t. $Q_U \otimes U \subset QH^*_U(X)$ is a Frobenius sub-algebra.

$\rho : \hat{H}^*(X,L) \longrightarrow H^*(X)$

2) $\gamma = \sum_{i=1}^{N} t_i \gamma_i$, $\{[\gamma_i]\}_{i=1}^{N}$ basis for $W := \rho^{-1}(U)$

3) $\int_L b \in \Lambda[[s_1, \ldots, s_M]]$.

Ex.: – $X$ CY3 $\Rightarrow$ $b \in A^1(L) \otimes R_W$ $\Rightarrow$ $\int_L b = 0 \in \Lambda[[s_j]]$

– $H^*(L;\mathbb{R}) \simeq H^*(S^n;\mathbb{R})$ $\Rightarrow$ $\exists!\, b$ s.t. $\int_L b = s$, $S = \mathrm{Span}\{s\}$.

**Theorem (Open WDVV equations).** Let $c$ be the coefficient of the Maurer-Cartan equation for the bounding pair $(\gamma_W, b)$, and let $u, v \in W \oplus S$, $w \in W$. Let $u_W, v_W$, denote the projections of $u, v$, to $W$, and let $\bar{w} = \rho(w), \bar{u} = \rho(u_W), \bar{v} = \rho(v_W)$. Then,

$$\sum_{l \in I_{W'}, m \in I_U} \partial_u \partial_l \overline{\Omega} \cdot g^{lm} \cdot \rho^* \partial_m \partial_{\bar{w}} \partial_{\bar{v}} \Phi - \partial_u c \cdot \partial_w \partial_v \overline{\Omega} =$$
$$= \sum_{l \in I_U, m \in I_{W'}} \rho^* \partial_{\bar{u}} \partial_{\bar{w}} \partial_l \Phi \cdot g^{lm} \cdot \partial_m \partial_v \overline{\Omega} - \partial_u \partial_w \overline{\Omega} \cdot \partial_v c.$$

follows from $2^{nd}$ component of Main Theorem.

**To define quantum product:**

$\hat{Q}_W := Q_W \otimes_{\Lambda_c} \Lambda$ $= \Lambda[[t_1, \ldots, t_N]]$

$\mathrm{Cone}(\hat{i} : A^*(X; \hat{Q}_W) \longrightarrow \hat{Q}_W)$ $H^*(\mathrm{Cone}(\hat{i})) = \hat{H}^*(X, L; \hat{Q}_W)$

$\mathrm{Cone}(\hat{\underline{i}} : A^*(X;\hat{Q}_W) \longrightarrow \hat{Q}_W)$

$\eta \longmapsto \int_L i^*\eta$

$H^*(\mathrm{Cone}(\hat{\underline{i}})) = \hat{H}^*(X, L; \hat{Q}_W)$.

$\cup$

$\hat{Q}_W \otimes W$

$\hat{\jmath} : \mathrm{Cone}(\hat{\underline{i}}) \longrightarrow \mathrm{Cone}(\hat{\underline{i}})$

$\rightsquigarrow \hat{\jmath} : H^*(\mathrm{Cone}(\hat{\underline{i}})) \longrightarrow H^*(\mathrm{Cone}(\hat{\underline{i}}))$

$QH^*_u(X,L) := \hat{Q}_W \otimes W$

$$\mathcal{Y} : QH^*_u(X,L)^{\otimes 2} \longrightarrow QH^*_u(X,L)$$

$$\mathcal{Y}(u,v) := \partial_u \hat{\jmath}(v).$$

<u>Theorem:</u> $\mathcal{Y}$ is commutative,
associative,
and invariant under gauge equivalence.

<u>Note:</u> Main Theorem $\Rightarrow$ assoc.

$$\mathcal{Y}(u, \mathcal{Y}(v,w)) = \partial_u \hat{\jmath}(\mathcal{Y}(v,w)) = \partial_u \hat{\jmath}(\partial_v \hat{\jmath}(w)) \overset{MT}{=} \partial_v \hat{\jmath} \circ \partial_u \hat{\jmath}(w)$$

$$= \mathcal{Y}(v, \mathcal{Y}(u,w))$$

$$\| \qquad\qquad = \mathcal{Y}(v, \mathcal{Y}(w,u))$$

$$\mathcal{Y}(u, \mathcal{Y}(w,v))$$

$\square$

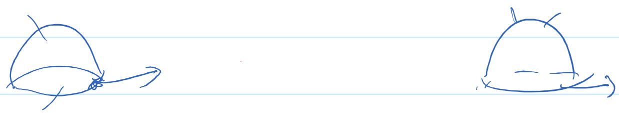
<u>Remark:</u>

**Theorem.** Suppose $[L] \neq 0$ and let $\eta \in H^*(X;\mathbb{R})$ such that $\int_L \eta = 1$. Then

$$c = \sum_{\substack{\beta \in H_2(X,\mathbb{Z}) \\ l \geq 0}} (-1)^{n+1+w_s(\beta)} \frac{T^{\varpi(\beta)}}{l!} \mathrm{GW}_\beta(\eta, PD([L]), [\gamma]^{\otimes l}).$$

**Theorem (Wall-crossing).** Suppose $[L] = 0$, and $\int_L b = s$ for $S = Span(s)$. Then

$$\overline{\mathrm{OGW}}_{\beta,k+1}(\eta_1,\ldots,\eta_l) = -\,\overline{\mathrm{OGW}}_{\beta,k}(\Gamma_\diamond,\eta_1,\ldots,\eta_l).$$



**Example:** $QH^*(\mathbb{CP}^n, \mathbb{RP}^n)$.

$U = H^*(X)$

$W = \hat{H}^*(X,L) \simeq H^*(X) \oplus \mathbb{R}[n+1]$

$$\begin{tikzcd} & W \arrow[dr, "P"] & \\ \mathbb{R} \arrow[ur] & & H^*(X) \arrow[ll, "i_{\mathbb{R}}"] \end{tikzcd}$$

basis: $\{\Gamma_j\}_{j=0}^{n} \cup \{\Gamma_\star\}$, $\quad \Gamma_j = [\omega_{FS}^{\wedge j}] \in H^*(X)$

$\Gamma_\star \in \mathbb{R}[n+1]$

$\Gamma_\star \in Ker(p)$ basis

$$\Gamma_i \star \Gamma_j = \sum_{\substack{0 \le m \le n \\ d \ge 0}} (-1)^{\frac{n+1}{2}d}\, T^d\, GW_d(\Gamma_i, \Gamma_j, \Gamma_m)\cdot \Gamma_{n-m} + \sum_{d \ge 0} T^{d/2}\, \overline{OGW}_{d,0}(\Gamma_i,\Gamma_j)\cdot \Gamma_\star$$

$$\underset{\text{deg axiom}}{=} \sum_{d=0,1} (-1)^{\frac{n+1}{2}d}\, T^d \cdot [T^d](\Gamma_i * \Gamma_j) + \sum_{d=0,1} T^{d/2}\, \underbrace{\overline{OGW}_{d,0}(\Gamma_i,\Gamma_j)}_{=\,?}\cdot \Gamma_\star$$

**d=0:** Zero axiom:

$$\overline{\mathrm{OGW}}_{\beta_0,k}(A_1,\ldots,A_l) = \begin{cases} -1, & (k,l) = (1,1) \text{ and } A_1 = 1, \\ P_{\mathbb{R}}(A_1 \smile A_2), & (k,l) = (0,2), \\ 0, & \text{otherwise.} \end{cases}$$

$\Rightarrow$ only nontrivial value is

$$\overline{OGW}_{0,0}(\Gamma_\star, 1) = 1.$$

$\Gamma_\star \star 1 = 1 \cdot T^0 \cdot \Gamma_\star$

$\underline{d=1:}$ if, WLOG, $i = \star$, then

deg $\Rightarrow |\Gamma_j| = n+1 \Rightarrow$ possible values: $\Gamma_\star \longleftrightarrow$ pt in $\mathcal{L}$ — direct calculation

$\Gamma_\star \circledast \Gamma_\star = 2 \cdot T^{1/2} \cdot \Gamma_\star$ $\qquad OGW_{1,0}(\Gamma_\star, \Gamma_\star) = OGW_{1,2} = 2,$

$\Gamma_\star \circledast \Gamma_{\frac{n+1}{2}} = 0$ $\qquad OGW_{1,0}(\Gamma_\star, \Gamma_{\frac{n+1}{2}}) = -OGW_{1,1}(\Gamma_{\frac{n+1}{2}}) \overset{\text{OWDVV}}{=} 0.$

if $i,j \in \{0, \dots, n\}$, nonzero value:

OWDVV $\Rightarrow$ $\quad OGW_{1,0}(\Gamma_1, \Gamma_n) = \frac{1}{2} OGW_{1,0}(\Gamma_n) = \frac{1}{2} \cdot (-1)^{\frac{n-1}{2}}.$

$$\Gamma_1 \circledast \Gamma_n = \underbrace{\tfrac{1}{2} T^{1/2} (-1)^{\frac{n-1}{2}} \cdot \Gamma_\star + (-1)^{\frac{n+1}{2}} T^{1}}$$

In total,

$q = T^{1/2}$, $\quad x \longleftrightarrow \omega_{FS}$, $\quad y \longleftrightarrow \Gamma_\star$

$$QH^*(X,L) \simeq \mathbb{R}[[q]][x,y] \Big/ \begin{array}{l} \underbrace{x^{n+1} = (-1)^{\frac{n+1}{2}} q^2 + (-1)^{\frac{n-1}{2}} \cdot \frac{1}{2} q y} \\ y^2 = 2 \cdot q y \\ x^{\frac{n+1}{2}} y = 0 \end{array}$$