Mirror Symmetry and Fukaya Categories of Singular Hypersurfaces

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Overview

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Motivation: Homological Mirror Symmetry (HMS)

Basic version: HMS is a conjecture relating the Fukaya category of a Kähler manifold Y to the category of coherent sheaves on a 'mirror' Kähler manifold $\check{Y}$ of the same dimension.

- This naïve story is a bit too simple.
- *Finding* a mirror $\check{Y}$ is difficult, sometimes impossible.
- Even mirrors of smooth varieties are often singular; sometimes they are of the 'wrong' dimension.
- For instance, the basic building-blocks for gluing approaches to mirror symmetry.
- For any mirror construction, HMS should be an involution.

Thus we need a notion of Fukaya categories of singular varieties.

Motivation (ctd.)

In this talk, we'll focus on singular *hypersurfaces* and *complete intersections*:

- In general we expect the singular variety to need extra data in order to define its Fukaya category, but this intrinsic geometry is difficult to understand.
- Some work on orbifold case using equivariance.
- Given a *smoothing* of the hypersurface, we have a nearby fiber which has a Fukaya category:
- This nearby category comes with extra algebraic data: *Seidel's natural transformation*
- The invariant cycles theorem suggests we *localize* with respect to this data to obtain the Fukaya category of the singular fiber.

Partially Wrapped Fukaya Categories

Suppose X is a Liouville manifold, and $F \subset \partial^\infty X$ is a closed subset, called the *stop*. Sylvan and Ganatra-Pardon-Shende (GPS) defined a category $\mathcal{W}(X, F)$:

- Objects are (possibly non-compact) exact cylindrical Lagrangians avoiding F;
- Roughly, morphisms are intersections between Lagrangians, plus positive Reeb chords between their boundaries at infinity that avoid the stop F;
- Actual definition uses *localization*, where we quotient the category by the cones of a collection of morphisms.

For instance given $f : X \to \mathbb{C}$, the category $\mathcal{W}(X, f)$ is defined to be the partially-wrapped Fukaya category of X stopped along $f^{-1}(-\infty) \subset \partial^\infty X$

Cap and Cup Functors

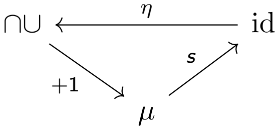
Given a Liouville hypersurface $F \subset \partial^\infty X$ we can take small *linking disks* which gives a functor

$$\cup : \mathcal{W}(F) \to \mathcal{W}(X, F)$$

The formal pullback on left Yoneda modules gives an adjoint functor

$$\cap : \mathrm{Mod} - \mathcal{W}(X, F) \to \mathrm{Mod} - \mathcal{W}(F)$$

The unit of the adjunction gives an exact triangle:



where s is Seidel's natural transformation (Abouzaid-Ganatra).

Definition

Definition (Auroux)

Suppose $f : X \to \mathbb{C}$ has precisely one singular fiber, lying over 0. Then the wrapped Fukaya category of $f^{-1}(0)$ is defined to be the localization of the wrapped Fukaya category of a nearby fiber $f^{-1}(t), t \neq 0$ at the natural transformation $s : \mu \to \mathrm{id}$:

$$D\mathcal{W}(f^{-1}(0)) = D\mathcal{W}(f^{-1}(t))[s^{-1}]$$

Lemma: this is equivalent to taking the quotient by the image of the $\cap$ functor.

Example

The basic example we'll consider throughout is the nodal conic $\{xy = 0\} \subset \mathbb{C}^2$. The smoothing is a cylinder $\{xy = t\}$, and the monodromy around $t = 0$ is given by a Dehn twist.

- The image of the cap functor in this case is an exact S^1, the vanishing cycling inside $\{xy = 1\}$.
- Under mirror symmetry, this corresponds to the point $1 \in \mathbb{C}^*$.
- Thus we have the expected mirror symmetry equivalence with the pair of pants.

Properties

- Works in a number of simple examples, very computable.
- Gives the expected Knörrer periodicity equivalence with a higher-dimensional LG model (Theorem 1)
- Gives the expected mirror symmetry equivalences for large complex structure limits (Theorem 2)
- Makes precise the mirror relationship between smoothing and compactification.
- Natural interpretation in terms of perverse schobers.
- Relation to other symplectic constructions such as Lagrangian cobordism groups, Viterbo restriction.
- Gives potentially interesting invariants of hypersurface singularities.
- Admits natural generalizations.

Knörrer Periodicity

Theorem (Orlov, Hirano)

If X is a smooth quasi-projective variety, and $f : X \to \mathbb{C}$ is a regular function, then there is an equivalence of categories

$$D^b\mathrm{Coh}(f^{-1}(0)) \to D^b\mathrm{Sing}(X \times \mathbb{C}, zf)$$

where z is the coordinate on $\mathbb{C}$.

- Note that X is smooth even when $f^{-1}(0)$ isn't.
- We could turn this theorem into a *definition* for the purposes of the A-model.
- Some work by Nadler already uses this as as a definition (using microlocal sheaves): uses $(\mathbb{C}^3, xyz)$ as mirror to the pair of pants.

Derived Knörrer Periodicity Theorem

Theorem (J)

Suppose $f : X \to \mathbb{C}$ is a regular (algebraic) function on a Stein manifold X having a single critical fiber $f^{-1}(0)$; then there is a quasiequivalence of A_∞-categories

$$D^\pi \mathcal{W}(f^{-1}(t))[s^{-1}] \to D^\pi \mathcal{W}(X \times \mathbb{C}, zf)$$

Derived Knörrer Periodicity Theorem

The proof goes via proving the equivalence in the smooth case:

Theorem (Abouzaid-Auroux-Katzarkov Equivalence)

Suppose $f : X \to \mathbb{C}$ is a regular function on a Stein manifold with a single critical fiber $f^{-1}(0)$; then when $t \neq 0$, we have a quasiequivalence of A_∞-categories:

$$T : \mathcal{W}(f^{-1}(t)) \to \mathcal{W}(X \times \mathbb{C}, z(f - t))$$

given by taking thimbles over admissible Lagrangians in the singular locus $f^{-1}(t)$.

Idea: all intersections and holomorphic curves are contained in the critical locus, around which we have a Morse-Bott neighbourhood. Needs to be made compatible with wrapping!

Proof Sketch

Once we have the equivalence in the smooth case:

$$T : \mathcal{W}(f^{-1}(t)) \to \mathcal{W}(X \times \mathbb{C}, z(f - t))$$

we can perform localization on both sides of the equivalence.

- passing from $(X \times \mathbb{C}, z(f - t))$ to $(X \times \mathbb{C}, zf)$ is a stop-removal,
- by the stop removal theorem of Sylvan, GPS, the category $\mathcal{W}(X \times \mathbb{C}, zf)$ may be obtained as a quotient of the category $\mathcal{W}(X \times \mathbb{C}, z(f - t))$ by linking disks,
- under the equivalence T, show that we quotient by the same thing, using a Künneth-type argument.

The theorem then follows.

Proof Sketch (ctd.)

Why is passing from $(X \times \mathbb{C}, z(f-t))$ to $(X \times \mathbb{C}, zf)$ a stop-removal?
Look at the geometry of the stop (the general fiber): changes from $X \setminus f^{-1}(t)$ to $X \setminus f^{-1}(0)$:

Theorem (J)

The Weinstein structure on $X \setminus f^{-1}(t)$ is obtained from $X \setminus f^{-1}(0)$ by attaching a collection of Weinstein handles.

The example of $(\mathbb{C}^2, xy)$ provides a nice illustration.

Proof Sketch (ctd.)

- We can explicitly identify the linking disks of these handles using GPS: they are exactly the functor $\cup$ applied to cocores ℓ of the handles.
- Finally, we can identify these linking disks with thimbles over $\cap$s using a Morse-Bott argument of Abouzaid-Smith:

Proposition

$$T(\cap\ell) \cong \bigcup \ell$$

Generalizations

Conjecture

Suppose $f : X \to \mathbb{C}$ is a regular function on a Stein manifold with a single critical fiber $f^{-1}(0)$ and suppose $g : X \to \mathbb{C}$ is another regular function: then we have a quasiequivalence for small $\delta > 0$

$$D^{\pi}\mathcal{W}(f^{-1}(0), g) \to D^{\pi}\mathcal{W}(X \times \mathbb{C}, zf + \delta g)$$

From which it should follow that:

Conjecture

Under appropriate hypotheses on $f_1, \ldots, f_k$, we have a quasiequivalence of A_∞-categories:

$$D^{\pi}\mathcal{W}(f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)) \simeq D^{\pi}\mathcal{W}(X \times \mathbb{C}^k, z_1 f_1 + \cdots + z_k f_k)$$

where $z_1, \ldots, z_k$ are coordinates on $\mathbb{C}^k$.

Applications to Mirror Symmetry

The fact that the Fukaya category depends on the choice of smoothing is a *feature* not a bug:

- Classically, the choice of the mirror depends on the entire degeneration
- The Gross-Siebert program suggests that this extra data should take the form of a *log structure* on $f^{-1}(0)$
- In good cases this is expected to determine a smoothing of $f^{-1}(0)$.
- perhaps an intrinsic construction using Parker's theory of holomorphic curves in *exploded manifolds*.
- the critical locus $f^{-1}(0) \times \mathbb{C}$ also comes with a (-1)-*shifted symplectic structure*
- perhaps an intrinsic construction using Joyce's theory of *d*-critical loci.

Localization

Suppose X is a smooth algebraic variety, L is a line bundle with a section s, and let $U = X \setminus s^{-1}(0)$.

Theorem

Let $s : L^{-1} \otimes (\cdot) \to \mathrm{id}$ be the natural transformation given by the section s. Then localizing at s gives an equivalence of categories:

$$D^b\mathrm{Coh}(X)[s^{-1}] \cong D^b\mathrm{Coh}(U)$$

Heuristic

Smoothing is mirror to compactifying.

Elliptic Curves

Consider the case of an elliptic curve with one node:

- the map $f : X \to \mathbb{C}$ given by the Tate family of elliptic curves gives a smoothing of $f^{-1}(0)$.
- the monodromy around 0 is given by a Dehn twist;
- we know HMS between the general fiber $f^{-1}(t)$ and a mirror elliptic curve E.
- the natural transformation $\mu \to \mathrm{id}$ is mirror to a section s of a degree-1 line bundle $\mathcal{L}$.

After localizing both sides we get the desired mirror symmetry equivalence:

Proposition

$$D^\pi \mathscr{F}(f^{-1}(0)) \simeq D^b\mathrm{Coh}(E \setminus \{p\})$$

Pairs of Pants

Higher-dimensional pair of pants are $\Pi_n = \{x_1 + \cdots + x_{n+1} + 1 = 0\} \subset (\mathbb{C}^*)^{n+1}$. Their mirrors are given by $\{z_1 \dots z_{n+1} = 0\} \subset \mathbb{C}^{n+1}$ with smoothing $\cong (\mathbb{C}^*)^n$

Theorem

We have quasiequivalences of categories:

$$D^\pi \mathcal{W}(\{z_1 \dots z_{n+1} = 0\}) \simeq D^\pi \mathcal{W}(\mathbb{C}^{n+2}, z_1 \dots z_{n+2}) \simeq D^b \mathrm{Coh}(\Pi_n)$$

The first category is given by the localization of the category of $\mathbb{C}[x_1^\pm, \dots, x_n^\pm]$-modules at the natural transformation $\mathrm{id} \to \mathrm{id}$ given by multiplication by $x_1 + \cdots + x_n + 1$. This is the same as the category of coherent sheaves on $\{x_1 + \cdots + x_n + 1 \neq 0\} \subset (\mathbb{C}^*)^n$, i.e. Π_n.

Large Complex Structure Limits

Theorem (J)

Suppose B is an integral affine manifold (without singularities), and let X and $\check{X}$ be the corresponding mirror pair. Suppose X and $\check{X}$ are homologically mirror via the family Floer construction of AGS; then the large complex structure limit X_0 of X is homologically mirror to the large volume limit of $\check{X}$:

$$D^{\pi}\mathscr{F}(X_0) \simeq D^b\mathrm{Coh}(\check{X} \setminus s^{-1}(0))$$

where $s^{-1}(0)$ is some divisor Poincaré dual to the Kähler form on $\check{X}$.

Proof Sketch

- Gross-Siebert's 'canonical section' $\sigma_1 : B \to X$ is mirror under the family Floer functor to the ample line bundle $\mathcal{L}$ defining the Kähler form on $\check{X}$.
- this is because the Legendre transform of the developing map gives exactly the tropical affine function on the mirror defining the Kähler form.
- under the family Floer functor, the fiberwise translation by a section σ_1 is mirror to tensoring by the mirror line bundle $\mathcal{L}$,
- and Seidel's natural transformation is mirror to multiplication by a section of $\mathcal{L}$.
- Now compare the localizations of both sides!

Thank you!