Mirror Symmetry and Fukaya Categories of Singular Hypersurfaces

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### Overview

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**Basic version:** HMS is a conjecture relating the Fukaya category of a Kähler manifold $\mathcal{Y}$ to the category of coherent sheaves on a ‘mirror’ Kähler manifold $\mathcal{Y}$ of the same dimension.

- This naïve story is a bit too simple.
- *Finding* a mirror $\mathcal{Y}$ is difficult, sometimes impossible.
- Even mirrors of smooth varieties are often singular; sometimes they are of the ‘wrong’ dimension.
- For instance, the basic building-blocks for gluing approaches to mirror symmetry.
- For any mirror construction, HMS should be an involution.

Thus we need a notion of Fukaya categories of singular varieties.
Motivation (ctd.)

In this talk, we’ll focus on singular *hypersurfaces* and *complete intersections*:

- In general we expect the singular variety to need extra data in order to define its Fukaya category, but this intrinsic geometry is difficult to understand.
- Some work on orbifold case using equivariance.
- Given a *smoothing* of the hypersurface, we have a nearby fiber which has a Fukaya category:
- This nearby category comes with extra algebraic data: *Seidel’s natural transformation*
- The invariant cycles theorem suggests we *localize* with respect to this data to obtain the Fukaya category of the singular fiber.
Suppose $X$ is a Liouville manifold, and $F \subset \partial^\infty X$ is a closed subset, called the *stop*. Sylvan and Ganatra-Pardon-Shende (GPS) defined a category $\mathcal{W}(X, F)$:

- Objects are (possibly non-compact) exact cylindrical Lagrangians avoiding $F$;
- Roughly, morphisms are intersections between Lagrangians, plus positive Reeb chords between their boundaries at infinity that avoid the stop $F$;
- Actual definition uses *localization*, where we quotient the category by the cones of a collection of morphisms.

For instance given $f : X \to \mathbb{C}$, the category $\mathcal{W}(X, f)$ is defined to be the partially-wrapped Fukaya category of $X$ stopped along $f^{-1}(-\infty) \subset \partial^\infty X$.
Given a Liouville hypersurface $F \subset \partial^\infty X$ we can take small *linking disks* which gives a functor

$$\cup : \mathcal{W}(F) \to \mathcal{W}(X, F)$$

The formal pullback on left Yoneda modules gives an adjoint functor

$$\cap : \text{Mod} - \mathcal{W}(X, F) \to \text{Mod} - \mathcal{W}(F)$$

The unit of the adjunction gives an exact triangle:

$$\cap \cup \xleftarrow{\eta} \text{id} \xrightarrow{s} +1 \xleftarrow{\mu}$$

where $s$ is Seidel’s natural transformation (Abouzaid-Ganatra).
Definition (Auroux)

Suppose $f : X \to \mathbb{C}$ has precisely one singular fiber, lying over 0. Then the wrapped Fukaya category of $f^{-1}(0)$ is defined to be the localization of the wrapped Fukaya category of a nearby fiber $f^{-1}(t)$, $t \neq 0$ at the natural transformation $s : \mu \to \text{id}$:

$$DW(f^{-1}(0)) = DW(f^{-1}(t))[s^{-1}]$$

**Lemma**: this is equivalent to taking the quotient by the image of the $\cap$ functor.
The basic example we’ll consider throughout is the nodal conic \( \{xy = 0\} \subset \mathbb{C}^2 \). The smoothing is a cylinder \( \{xy = t\} \), and the monodromy around \( t = 0 \) is given by a Dehn twist.

- The image of the cap functor in this case is an exact \( S^1 \), the vanishing cycling inside \( \{xy = 1\} \).
- Under mirror symmetry, this corresponds to the point \( 1 \in \mathbb{C}^* \).
- Thus we have the expected mirror symmetry equivalence with the pair of pants.
Properties

- Works in a number of simple examples, very computable.
- Gives the expected Knörrer periodicity equivalence with a higher-dimensional LG model (Theorem 1)
- Gives the expected mirror symmetry equivalences for large complex structure limits (Theorem 2)
- Makes precise the mirror relationship between smoothing and compactification.
- Natural interpretation in terms of perverse schobers.
- Relation to other symplectic constructions such as Lagrangian cobordism groups, Viterbo restriction.
- Gives potentially interesting invariants of hypersurface singularities.
- Admits natural generalizations.
Knörrer Periodicity

Theorem (Orlov, Hirano)

If $X$ is a smooth quasi-projective variety, and $f : X \to \mathbb{C}$ is a regular function, then there is an equivalence of categories

$$D^b \text{Coh}(f^{-1}(0)) \to D^b \text{Sing}(X \times \mathbb{C}, zf)$$

where $z$ is the coordinate on $\mathbb{C}$.

- Note that $X$ is smooth even when $f^{-1}(0)$ isn’t.
- We could turn this theorem into a definition for the purposes of the A-model.
- Some work by Nadler already uses this as a definition (using microlocal sheaves): uses $(\mathbb{C}^3, xyz)$ as mirror to the pair of pants.
**Theorem (J)**

Suppose $f : X \to \mathbb{C}$ is a regular (algebraic) function on a Stein manifold $X$ having a single critical fiber $f^{-1}(0)$; then there is a quasiequivalence of $A_\infty$-categories

$$D^\pi \mathcal{W}(f^{-1}(t))[s^{-1}] \to D^\pi \mathcal{W}(X \times \mathbb{C}, zf)$$
The proof goes via proving the equivalence in the smooth case:

**Theorem (Abouzaid-Auroux-Katzarkov Equivalence)**

Suppose $f : X \to \mathbb{C}$ is a regular function on a Stein manifold with a single critical fiber $f^{-1}(0)$; then when $t \neq 0$, we have a quasiequivalence of $A_\infty$-categories:

$$T : \mathcal{W}(f^{-1}(t)) \to \mathcal{W}(X \times \mathbb{C}, z(f - t))$$

given by taking thimbles over admissible Lagrangians in the singular locus $f^{-1}(t)$.

Idea: all intersections and holomorphic curves are contained in the critical locus, around which we have a Morse-Bott neighbourhood. Needs to be made compatible with wrapping!
Proof Sketch

Once we have the equivalence in the smooth case:

\[ T : \mathcal{W}(f^{-1}(t)) \rightarrow \mathcal{W}(X \times \mathbb{C}, z(f - t)) \]

we can perform localization on both sides of the equivalence.

- passing from \((X \times \mathbb{C}, z(f - t))\) to \((X \times \mathbb{C}, zf)\) is a stop-removal,
- by the stop removal theorem of Sylvan, GPS, the category \(\mathcal{W}(X \times \mathbb{C}, zf)\) may be obtained as a quotient of the category \(\mathcal{W}(X \times \mathbb{C}, z(f - t))\) by linking disks,
- under the equivalence \(T\), show that we quotient by the same thing, using a Küneth-type argument.

The theorem then follows.
Why is passing from \((X \times \mathbb{C}, z(f - t))\) to \((X \times \mathbb{C}, zf)\) a stop-removal? Look at the geometry of the stop (the general fiber): changes from \(X \setminus f^{-1}(t)\) to \(X \setminus f^{-1}(0)\):

**Theorem (J)**

*The Weinstein structure on \(X \setminus f^{-1}(t)\) is obtained from \(X \setminus f^{-1}(0)\) by attaching a collection of Weinstein handles.*

The example of \((\mathbb{C}^2, xy)\) provides a nice illustration.
We can explicitly identify the linking disks of these handles using GPS: they are exactly the functor $\cup$ applied to cocores $\ell$ of the handles.

Finally, we can identify these linking disks with thimbles over $\cap$s using a Morse-Bott argument of Abouzaid-Smith:

$$T(\cap \ell) \cong \bigcup \ell$$
Suppose $f : X \to \mathbb{C}$ is a regular function on a Stein manifold with a single critical fiber $f^{-1}(0)$ and suppose $g : X \to \mathbb{C}$ is another regular function: then we have a quasiequivalence for small $\delta > 0$

$$D^\pi \mathcal{W}(f^{-1}(0), g) \to D^\pi \mathcal{W}(X \times \mathbb{C}, zf + \delta g)$$

From which it should follow that:

Under appropriate hypotheses on $f_1, \ldots, f_k$, we have a quasiequivalence of $A_\infty$-categories:

$$D^\pi \mathcal{W}(f_1^{-1}(0) \cap \cdots \cap f_k^{-1}(0)) \simeq D^\pi \mathcal{W}(X \times \mathbb{C}^k, z_1 f_1 + \cdots + z_k f_k)$$

where $z_1, \ldots, z_k$ are coordinates on $\mathbb{C}^k$. 
Applications to Mirror Symmetry

The fact that the Fukaya category depends on the choice of smoothing is a \textit{feature} not a bug:

- Classically, the choice of the mirror depends on the entire degeneration.
- The Gross-Siebert program suggests that this extra data should take the form of a \textit{log structure} on $f^{-1}(0)$.
- In good cases this is expected to determine a smoothing of $f^{-1}(0)$.
- Perhaps an intrinsic construction using Parker’s theory of holomorphic curves in \textit{exploded manifolds}.
- The critical locus $f^{-1}(0) \times \mathbb{C}$ also comes with a \textit{$(-1)$-shifted symplectic structure}.
- Perhaps an intrinsic construction using Joyce’s theory of $d$-critical loci.
Suppose $X$ is a smooth algebraic variety, $L$ is a line bundle with a section $s$, and let $U = X \setminus s^{-1}(0)$.

**Theorem**

Let $s : L^{-1} \otimes (\cdot) \to \text{id}$ be the natural transformation given by the section $s$. Then localizing at $s$ gives an equivalence of categories:

$$D^b\text{Coh}(X)[s^{-1}] \cong D^b\text{Coh}(U)$$

**Heuristic**

Smoothing is mirror to compactifying.
Consider the case of an elliptic curve with one node:

- the map \( f : X \to \mathbb{C} \) given by the Tate family of elliptic curves gives a smoothing of \( f^{-1}(0) \).
- the monodromy around 0 is given by a Dehn twist;
- we know HMS between the general fiber \( f^{-1}(t) \) and a mirror elliptic curve \( E \).
- the natural transformation \( \mu \to \text{id} \) is mirror to a section \( s \) of a degree-1 line bundle \( L \).

After localizing both sides we get the desired mirror symmetry equivalence:

**Proposition**

\[
D^\pi \mathcal{F}(f^{-1}(0)) \sim D^b \text{Coh}(E \setminus \{p\})
\]
Higher-dimensional pair of pants are
\[ \Pi_n = \{ x_1 + \cdots + x_{n+1} + 1 = 0 \} \subset (\mathbb{C}^*)^{n+1} \]. Their mirrors are given by
\[ \{ z_1 \cdots z_{n+1} = 0 \} \subset \mathbb{C}^{n+1} \] with smoothing \( \cong (\mathbb{C}^*)^n \).

**Theorem**

We have quasiequivalences of categories:

\[ D^\pi \mathcal{W}(\{z_1 \cdots z_{n+1} = 0\}) \cong D^\pi \mathcal{W}(\mathbb{C}^{n+2}, z_1 \cdots z_{n+2}) \cong D^b \text{Coh}(\Pi_n) \]

The first category is given by the localization of the category of \( \mathbb{C}[x_1^\pm, \ldots, x_n^\pm] \)-modules at the natural transformation \( \text{id} \to \text{id} \) given by multiplication by \( x_1 + \cdots + x_n + 1 \). This is the same as the category of coherent sheaves on \( \{ x_1 + \cdots + x_n + 1 \neq 0 \} \subset (\mathbb{C}^*)^n \), i.e. \( \Pi_n \).
Theorem (J)

Suppose $B$ is an integral affine manifold (without singularities), and let $X$ and $\check{X}$ be the corresponding mirror pair. Suppose $X$ and $\check{X}$ are homologically mirror via the family Floer construction of AGS; then the large complex structure limit $X_0$ of $X$ is homologically mirror to the large volume limit of $\check{X}$:

$$D^\pi \mathcal{F}(X_0) \simeq D^b\text{Coh}(\check{X} \setminus s^{-1}(0))$$

where $s^{-1}(0)$ is some divisor Poincaré dual to the Kähler form on $\check{X}$.
Proof Sketch

- Gross-Siebert’s ‘canonical section’ $\sigma_1 : B \to X$ is mirror under the family Floer functor to the ample line bundle $\mathcal{L}$ defining the Kähler form on $\tilde{X}$.

- this is because the Legendre transform of the developing map gives exactly the tropical affine function on the mirror defining the Kähler form.

- under the family Floer functor, the fiberwise translation by a section $\sigma_1$ is mirror to tensoring by the mirror line bundle $\mathcal{L}$,

- and Seidel’s natural transformation is mirror to multiplication by a section of $\mathcal{L}$.

- Now compare the localizations of both sides!
Thank you!