Sutured Legendrian homologies and applications to the conormal construction

Côme Dattin, Uppsala University

Symplectic Zoominar

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**Sutured contact mfd** [Colin-Ghiggini-Honda-Hutchings]

\((V, \lambda)\) with corners.

\((W_{\pm}, \beta_{\pm})\) Liouville domains, of contact boundary \((\Gamma, \lambda_\Gamma)\)

\(\simeq\) contact mfd with smooth convex boundary [Giroux]

**Sutured Legendrian submanifold**

\(\Lambda \subset (V, \lambda)\) such that

- \(\partial \Lambda \subset \{0\} \times \Gamma\);
- near the vertical boundary,
  \(\Lambda \cong (-\epsilon, 0] \times \{0\} \times \partial \Lambda\).

\(\simeq \partial \Lambda \subset \Gamma\) Legendrian

*Example*: Legendrian lift of an exact cylindrical Lagrangian filling
Completion

[CGHH] \((V, \lambda)\) extended into a non-compact contact mfd \(V^*\).
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\[ \Gamma \Rightarrow \Lambda_0 \Rightarrow \Lambda_1 \Rightarrow (W_+, \beta_+) \]

[Abbondandolo-Schwarz]

- **Cylindrical** completion: \(\Lambda^* = \Lambda \cup (\mathbb{R}_\tau^+ \times \{0\} \times \partial \Lambda)\)
- **Wrapped** completion: a Hamiltonian quadratic in \(\tau\) induces a contact v.f.
  \(\rightsquigarrow \Lambda^W = \phi^1(\Lambda^*)\).

\{chords of \(\Lambda_0^W \cup \Lambda_1^*\) outside the original mfd\} \(\leftrightarrow\) \{chords of \(\Gamma\) from \(\partial \Lambda_0\) to \(\partial \Lambda_1\)\}
Relative sutured Legendrian homologies [Chekanov, Ekholm-Etnyre-Sullivan...]  
\( \Lambda_0, \Lambda_1 \subset (V, \lambda) \) sutured Legendrians, hypertight  
- \( LC(\Lambda_0, \Lambda_1, V, \lambda) \) generated by Reeb chords going from \( \Lambda_0^* \) to \( \Lambda_1^* \)  
- \( WLC(\Lambda_0, \Lambda_1, V, \lambda) \) ... from \( \Lambda_0^W \) to \( \Lambda_1^* \).  

Well-defined (maximum principle)  

**Theorem**  
\( LC(\Lambda_0, \Lambda_1, V) \) is a sub-complex of \( WLC(\Lambda_0, \Lambda_1, V) \), inducing an exact sequence  
\[
\rightarrow LH(\Lambda_0, \Lambda_1, V) \rightarrow W LH(\Lambda_0, \Lambda_1, V) \rightarrow LH^{\text{ext}}(\Lambda_0, \Lambda_1, V) \rightarrow \]

\( LC^{\text{ext}} \) generated by the exterior chords (ie from \( \partial \Lambda_0 \) to \( \partial \Lambda_1 \) in \( \Gamma \))  

Homologies are invariants along Legendrian paths, where the boundary can move.  
[Dimitroglou Rizell] Generalise Lagrangian Floer homology (via Legendrian lift)  

**Expectations** : The exact sequence is invariant along paths with fixed boundary.  

**Remark** (Seidel isomorphism [Ekholm, Dimitroglou Rizell]) :  
\( LH^{\text{ext}}(\Lambda_0, \Lambda_1, V) \) should be a bilinearized version of the dga \( LC(\partial \Lambda, \Gamma) \)
The conormal of braids

Conormal construction

**Theorem**

\[ N \subset M, \text{ such that } \partial N \subset \partial M \]
\[ \Rightarrow U_N M \subset U M \text{ sutured Legendrian} \]

\( \Rightarrow \) invariants of smooth submanifolds with boundary

**Theorem**

The sutured exact sequence is a complete invariant for local 2-braids.

\( \Rightarrow \) If the conormals of two local 2-braids are Legendrian isotopic with **fixed boundary**, the braids are equivalent.

**Remark** [Shende, Ekholm-Ng-Shende] : For a knot \( K \subset S^3 \), the torus \( U_K S^3 \) is a complete knot invariant.
Apply the conormal construction to a pure local 2-braid $B^k \subset [-1,1]_u \times \Sigma$

$V = U\left([-1,1] \times \Sigma\right)$

$\simeq [-1,1] \times (D\Sigma \cup \Sigma D\Sigma)$

$\Lambda^k = U_{B^k}\left([-1,1] \times \Sigma\right)$ two Legendrian cylinders

contact manifold with smooth convex boundary
The conormal of braids

The sutured manifold

Metric \( \frac{du^2 + g_\Sigma}{1 + u^2} \) on \([-1, 1]u \times \Sigma \)

\( \rightsquigarrow \) contact form on the unit bundle.

Reeb trajectories project to geodesics.

"hyperbolic" metric

geodesic

\( D\Sigma \quad U\Sigma \quad D\Sigma \)

downward vectors

upward vectors

\( \Lambda \)
The conormal of braids

The sutured manifold

Metric $\frac{du^2 + g\Sigma}{1 + u^2}$ on $[-1, 1] \times \Sigma$

$\sim$ contact form on the unit bundle.
Reeb trajectories project to geodesics.

$W_+ = W_- = D\Sigma \sqcup D\Sigma$
The conormal of braids

**The sutured complex**

Reeb trajectories $\leftrightarrow$ geodesics $\Rightarrow$ hypertight

$\partial \Lambda = \text{fibers} \subset (U \Sigma, \lambda_\Sigma)$

$\forall \gamma \in \pi_1(\Sigma)$ we get:
- 1 interior chord ($u = 0$)
- 2 exterior chords ($u > 1$ and $u < -1$)

Split by homotopy class.

The $\mathbb{Z}_2[H_1(\Lambda_0)]$-$\mathbb{Z}_2[H_1(\Lambda_1)]$-bimodule with one generator is $C = \bigoplus \mathbb{Z}_2 \mu_0^i c \mu_1^j$

$$LC(\Lambda) = C^0 \quad \mathcal{W}LC(\Lambda) = C^{-}[1] \oplus C^0 \oplus C^+[1]$$
Lifting to 1-jets space

Lift $\Sigma$ to $\mathbb{R}^2$, and use $U\mathbb{R}^2 \simeq J^1(S^1_{\theta})$. After completion, $\Lambda^k$ lifts to $J^1(\mathbb{R}_u \times S^1_{\theta})$ (similar to [Pan-Rutherford])

Reeb chords correspond to positive critical points of $f_k$.

[Floer, Ekholm] Holomorphic curves degenerate to gradient trajectories 

$\rightsquigarrow \partial c^- = c^0$ and $\partial c^+ = \mu_0^{-k} c^0 \mu_1^k$
The conormal of braids

Sutured exact sequence: In homology we obtain

\[ C^0 \xrightarrow{0} H(C^+ \oplus C^0 \oplus C^-) \xrightarrow{f_k} C^- \oplus C^+ \xrightarrow{\text{Id} \oplus \delta_k} C^0 \xrightarrow{0} \]

where \( \delta_k c = \mu_0^{-k} c \mu_1^k \).

By contradiction: Assume \( \Lambda^0 \sim \Lambda^k \) as Legendrians with fixed boundaries \( (k \neq 0) \)

Homotopic and grading restrictions \( \Rightarrow \)

Impossible: \( f_0(c) = (c, c) \notin \ker(\text{Id} \oplus \delta_k) \).

Conclusion: \( \Lambda^k \sim \Lambda^l \Rightarrow k = l \)
The end.

Thank you for your attention