Capacities from the Chiu-Tamarkin Complex.

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2006: Elishberg–Kim–Poltervich:

[$B_A \times S^1 \underset{\text{contact}}{\not\hookrightarrow} B_a \times S^1$ if $a < A$ and $\exists\, m \in \mathbb{Z}_{>0}$ s.t. $m \in [a, A]$]

idea: compare contact homology.

e.g. $A \geq 1$

2017: Chiu: [It is still true if $\exists\, m \in \mathbb{Z}_{>0}$, and $m \leq a < A \leq m+1$]

idea: compare the Chiu–Tamarkin complex of balls, a sheaf theoritic inv. similar to contact homology.

(2015: also by Fraser using $\mathbb{Z}/\ell\mathbb{Z}$-equi. contact homology).

Our work: Extend the construction of Chiu in two aspects:

1) Chiu-Tamarkin cpx of more open sets.

$\rightsquigarrow$ Capacities.

2) Computation of Chiu-Tamarkin cpx for convex toric domains.

$\rightsquigarrow$ capacities of convex toric domain.

$D(X\times\mathbb{R}) :=$ derived cate. of $\mathbb{K}$-v.s. sheaves over $X\times\mathbb{R}$

For $F\in D(X\times Y\times\mathbb{R}_t)$, $G\in D(Y\times Z\times\mathbb{R}_t)$.

sheaf convolution

$$F * G := Rs_!(q_1^{-1}F\otimes q_2^{-1}G) \sim \int_{Y\times\mathbb{R}_s} F(x,y,s)\,G(y,z,t-s)\,dy\,ds$$

$$\begin{tikzcd} & X\times Y\times\mathbb{R}_{t_1} \\ X\times Y\times Z\times\mathbb{R}_1\times\mathbb{R}_2 \arrow[ur, "q_1"] \arrow[rr, "s"] \arrow[dr, "q_2"'] & & X\times Z\times\mathbb{R}_{t_1+t_2} \\ & Y\times Z\times\mathbb{R}_{t_2} \end{tikzcd}$$

Tamarkin Category $\mathcal{D}(X)$: Essential image of the convolution functor: $F\mapsto \mathbb{K}_{[0,+\infty)} * F$, in $D(X\times\mathbb{R})$

It is a triangulated category.

Let $\mu s(F) = \left\{ (q,p) \in T^*X : \begin{array}{l} \exists t \in \mathbb{R} \text{ s.t.} \\ (q,p,t,1) \in SS(F) \end{array} \right\}$

For an open set $U \subseteq T^*X$, consider

$$\begin{tikzcd} {}^{\perp}(\mathcal{D}_{U^c}(X)) \arrow[d, phantom, "\shortparallel"] & & \{F \in \mathcal{D}(X) : \mu s(F) \subseteq U^c\} \arrow[d, phantom, "\shortparallel"] \\ \mathcal{D}_U(X) \arrow[r, hook, shift left] & \mathcal{D}(X) \arrow[l, shift left, "*P_U"] \arrow[r, shift right, "*Q_U"'] & \mathcal{D}_{U^c}(X) \arrow[l, hook', shift right] \end{tikzcd}$$

right adjoint — left adjoint

We say U is **admissible** if two adjoints are convolution functors s.t. kernels (P_U, Q_U) fit into a d.t. :

$$P_U \longrightarrow \mathbb{K}_{\Delta_{X^2} \times [0,\infty)} \longrightarrow Q_U \xrightarrow{+1} \text{ in } D(X \times X \times \mathbb{R})$$

Prop. 1) bounded open sets and all toric domains are admissible.

2) If $U \overset{i}{\subseteq} V$, $\exists\ P_U \longrightarrow P_V$, $Q_U \to Q_V$, functorially. s.t. if $U = V$, the they are identity.

In particular, (P_U, Q_U) are unique.

For $U \subseteq T^*X \times S^1$. Lift it to a $\mathbb{Z}$-equi. open set $\tilde{U} \subseteq J^1 X$.

Let $\mu s_L(F) = \{(q,p,t): (q,p,t,1) \in SS(F)\} \subseteq J^1 X$

One can also disuss $\mathcal{D}_{U^c}$, $\mathcal{D}_U$

$\Rightarrow$ Apply to admissibility of contact $U \subseteq T^* X \times S^1$.

For $l \in \mathbb{Z}_{>0}$

$P_u^{\otimes l}$ over

Consider $$X^{2l} \times \mathbb{R}^l \xleftarrow{\tilde{\Delta}} X^l \times \mathbb{R}^l \xrightarrow{\pi} \mathbb{R}^l \xrightarrow{s} \mathbb{R}$$

$\tilde{\Delta}(q_1, \dots, q_n) = (q_l\ q_1\ q_1\ q_2\ \cdots\ q_{l-1}\ q_l)$

$s(t_1, \dots, t_n) = t_1 + \dots + t_l$, π projection.

Let $$F_l(U, \mathbb{K}) = Rs_! R\pi_! (\tilde{\Delta}^{-1} P_u^{\boxtimes l}) \text{ over } \mathbb{R}$$

$F_l(U, \mathbb{K})$ is a $\mathbb{Z}/l\mathbb{Z}$-equivariant sheaf.

via the cyclic permutation on factors.

$\mathbb{K}$-vector space : $F_l(U, \mathbb{K}) \cong F_1(U, \mathbb{K})$.

$(F_\ell(U,\mathbb{K}))_T = R\Gamma_c(\text{ discrete Hamiltonian loops, with action } \leq T\}$

The Chiu-Tamarkin complex is

$$C_{T,\ell}(U,\mathbb{K}) := R\mathrm{Hom}_{\mathbb{Z}/\ell\mathbb{Z}}\left(\left(F_\ell(U,\mathbb{K})\right)_T, \mathbb{K}[-\dim X]\right)$$

Here $T \geq 0$, $\ell \in \mathbb{Z}_{>0}$, $\mathbb{K}$ is the trivial $\mathbb{Z}/\ell\mathbb{Z}$-rep.

e.g. $H^* C_{T,\ell}(T^*X, \mathbb{K}) = \mathrm{Ext}^*_{\mathbb{Z}/\ell\mathbb{Z}}(\mathbb{K},\mathbb{K}) \otimes H^*(X,\mathbb{K})$,

$$H^* C_{T,p}(E(a_1,\dots,a_d),\mathbb{F}_p) = u^{-I(T/\bar{a})} \cdot \mathbb{F}_p[u] \otimes \Lambda(\theta)$$

where $|u|=2$, $|\theta|=1$, $p\geq 3$, $I(T/\bar{a}) = \lfloor \frac{T}{a_1} \rfloor + \lfloor \frac{T}{a_2} \rfloor + \dots + \lfloor \frac{T}{a_d} \rfloor$

$0 \leq T < p\, a_1$

Guillermou-Kashiwara-Schapira sheaf quantization of Hamiltonian isotopy $\Rrightarrow$

Thm (Chiu 2017) $C_{T,\ell}(U, \mathbb{K})$ is invariant under cpt supported Hamilton isotopies of T^*X for $U \subseteq T^*X$ and $T \geq 0$, $\ell \in \mathbb{Z}_{>0}$.

And $C_{T,\ell}(U, \mathbb{K})$ is invariant under cpt supported contact isotopies of $T^*X \times S^1$ for $U \subseteq T^*X \times S^1$, $\ell \in \mathbb{Z}_{>0}$, $T = \ell$.

Capaties

For $U \subseteq T^*\mathbb{R}^d$.

consider $P_U \to P_{T^*\mathbb{R}^d} = \mathbb{K}_{\Delta(\mathbb{R}^d)^2 \times [0,\infty)}$

$$\rightsquigarrow \quad F_\ell(U, \mathbb{K}) \longrightarrow F_\ell(T^*\mathbb{R}^d, \mathbb{K})$$

$$\xrightarrow{(-)_T,\ T\geq 0} \quad \eta_{T,\ell}(U,\mathbb{K}) = \left[(F_\ell(U,\mathbb{K}))_T \to \mathbb{K}[-d] \right]$$

$$\in H^0 C_{T,\ell}(U,\mathbb{K}).$$

Recall, $H^* C_{T,\ell}(U,\mathbb{K}) = \mathrm{Ext}^*_{\mathbb{Z}/\ell\mathbb{Z}}((F_\ell(U,\mathbb{K}))_T, \mathbb{K}[-d])$

is a right $A = \mathrm{Ext}^*_{\mathbb{Z}/\ell\mathbb{Z}}(\mathbb{K},\mathbb{K})$-module.

So one can assume $\mathrm{char}(\mathbb{K}) \mid \ell$, ℓ odd.

Then $A = \mathrm{Ext}^*_{\mathbb{Z}/\ell\mathbb{Z}}(\mathbb{K}, \mathbb{K}) = \mathbb{K}[u] \otimes \Lambda(\theta)$, $|u| = 2$, $|\theta| = 1$.

For ℓ odd, let P_ℓ = the minimal prime factor of ℓ.

Def

$$\boxed{\begin{array}{l} \mathrm{Spec}(U, k) = \{ T \geq 0,\ \exists\, \ell_0 \text{ odd, s.t. if all } \ell \text{ odd},\ P_\ell \geq P_{\ell_0},\ \eta_{T,\ell}(U, \mathbb{F}_{P_\ell}) = u^k \gamma \} \\ c_k(U) := \inf \mathrm{Spec}(U, k). \end{array}}$$

RMK: The definition similar to the Gutt-Huchtings one c_k^{GH} defined by S^1-equi. SH.

Symplectic situation

Thm (Z.) The functions $c_k : \{\text{admissible open sets}\} \to (0, +\infty]$

Then i) $c_k(U) \le c_{k+1}(U)$; ii) $c_k(U) \le c_k(V)$ if $U \subseteq V$.

iii) c_k is inv. under cpt support support Hamiltonian isotopy.

iv) If U is bounded with RCT boundary.

Then $\mu s(F_k(U)) \subseteq A(\partial U)$ ⇝ action spectrum.

Consequently, $\begin{cases} \text{if } A(\partial U) \text{ discrete. Then } c_k(U) \in A(\partial U). \\ c_k(rU) = r^2 c_k(U). \end{cases}$

i.e. $c_k : \{U \subseteq T^*\mathbb{R}^n : U \text{ bd}, \ \partial U \text{ RCT}\} \to (0, +\infty]$

is a symplectic capacities.

Contact situation

Similarly, for $U \subseteq T^*\mathbb{R}^d \times S^1$,

we define $[\mathrm{Spec}](U,k) = \mathrm{Spec}(U,k) \cap \{\text{odd numbers}\}$.

$$[c]_k(U) := \inf [\mathrm{Spec}](U,k)$$

Then i) ii) Still true.

And iii) is true for cpt support contactomorphism.

But iv) doesn't make much sense.

Convex toric domain

For $\Omega \overset{\text{open}}{\subseteq} \mathbb{R}^d_{\geq 0}$, s.t. $\hat{\Omega} = \{x : (|x_i|) \in \Omega\}$ is convex

Then we say $X_\Omega = \{u \in \mathbb{C}^d : (\pi|u_1|^2, \cdots, \pi|u_d|^2) \in \Omega\}$

to be a convex toric domain.

prop. (7.1) There is a generating function model of P_{X_Ω}.

$\Rightarrow$ Computing $c_{T,\ell}(X_\Omega, \mathbb{K})$ combinatorially.

Let $\Omega^\circ_T = \{z : T + \langle z, \zeta \rangle \geq 0, \forall \zeta \in \Omega\}$.

$I(z) = \sum_i \lfloor -z_i \rfloor$, $\quad I(\Omega^\circ_T) = \max_{z \in \Omega^\circ_T} I(z)$.

$\|\Omega^\circ_T\|_\infty = \max_{z \in \Omega^\circ_T} \|z\|_\infty$.

Thm (Z.) If $0 \le T < p^{\ell}/\|\Omega_1^\circ\|_\infty$, $X_\Omega \ne \mathbb{C}^d$ we have

- $H^* C_{T,\ell}(X_\Omega, \mathbb{F}_{p^\ell}) \cong H^{\geq -2I(\Omega_T^\circ)} C_{T,\ell}(X_\Omega, \mathbb{F}_{p^\ell})$,

and $H^{-2I(\Omega_T^\circ)} C_{T,\ell}(X_\Omega, \mathbb{F}_{p^\ell}) \ne 0$.

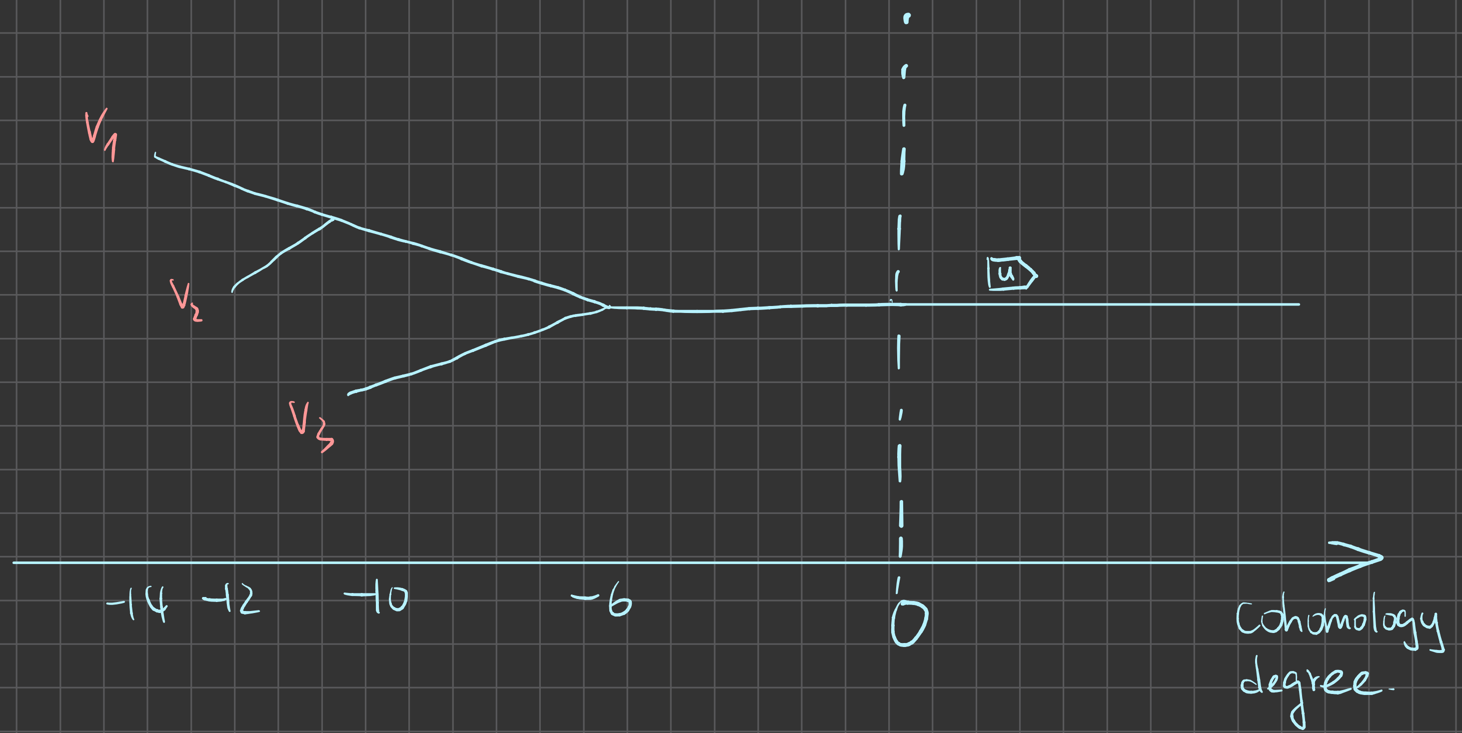
- $H^* C_{T,\ell}(X_\Omega, \mathbb{F}_{p^\ell})$, as $\mathbb{F}_{p^\ell}[u]$ module, it is of $rk = 2$

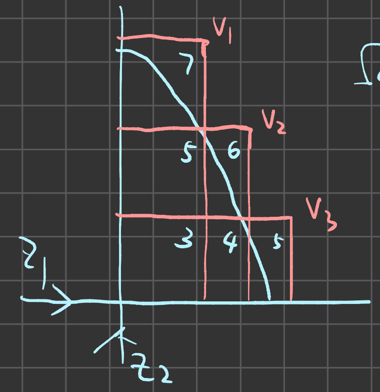
and its torsion part is in $[-2I(\Omega_T^\circ), -1]$.

- For $z \in \Omega_T^\circ$, $\exists$ decomposition $\eta_{T,\ell}(X_\Omega, \mathbb{F}_{p^\ell}) = u^{I(z)} \gamma_z$

s.t. γ_z is non-torsion.

Graphically, $H^* C_{T,\ell}(X_\Omega, \mathbb{F}_p)$





Coro $\quad c_k(X_\Omega) = \inf\{T : \exists z \in \Omega_T^\circ,\ I(z) \geq k\}$

And $\quad [c]_k(X_\Omega \times S^1) = \inf\left\{\ell \text{ odd} : \begin{array}{l} \frac{\ell}{p_\ell} < \frac{1}{\|\Omega_1^\circ\|_\infty}, \\ \exists z \in \Omega_\ell^\circ,\ I(z) \geq k \end{array}\right\}$

Consequently: $c_k(X_\Omega) = c_k^{GH}(X_\Omega)$, X_Ω convex toric domain.

Question: $c_k(U) = c_k^{GH}(U) = c_k^{EH}(U)$ in general?

(U convex or in general).

Or more general, in what sense $H^*C_{T,\ell}$ is related to $SH^*_{S^1}$?

Many evidences have been observed.

Thanks for your attention !