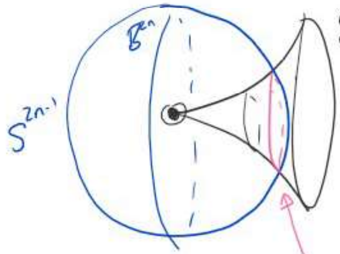
Unexpected fillings

Monday, February 8, 2021 12:12 PM

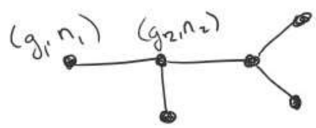
Joint with Olga Plamenevskaya

See Olga's "Western Hemisphere Symplectic Seminar" talk in a few hours for different focus points

Isolated Complex surface singularity



Link = $S \cap S^{2n-1}$ ← 3-manifold, contact structure



Smoothings (Milnor fibers)	Resolution	Symplectic fillings
Deform complex equations to nonsingular zero locus	Blow-up to normal crossing divisor with plumbing neighborhood	smooth symplectic 4-manifold with contact boundary the same as the link
↑ hard to classify in general	↑ well understood	↑ hard to classify in general

smoothings and resolutions are special cases of symplectic fillings

Question: Given an isolated complex surface singularity, does its link have symplectic fillings which are different from smoothings or resolutions?

(not diffeo or symplecto

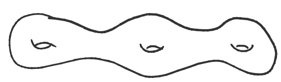
Prior results:

- Simple singularities [Ohta-Ono '03]
- elliptic → Simple elliptic singularities [Ohta-Ono '05]
- Cyclic singularities [Lisca '08, Nemethi-Popescu-Pampu '10]
- Quotient singularities [Bhupal-Ono '12, Park-Park-Shin-Urzúa '18]

rational

all symplectic fillings are diffeo to resolution or a smoothing

Non-rational examples [Akhmedov-Ozbagci '14]



∞ly many symp fillings with $b_1 > 0$ ← not smoothings

Our focus today:

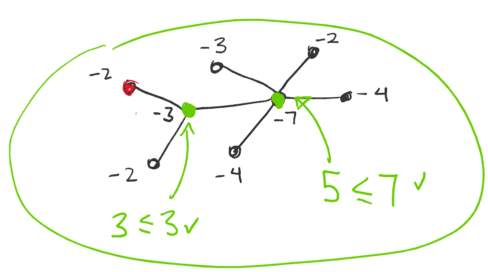
Rational surface singularities with reduced fundamental cycle (RFC)

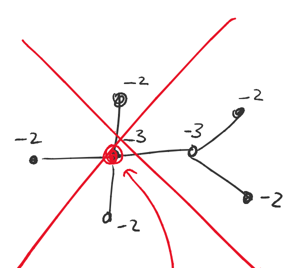
← low complexity from alg geom standpoint →

Resolution graph a tree of genus 0 components with

$$\boxed{\text{valence}(v) \le -v \cdot v}$$

$\forall$ vertices v

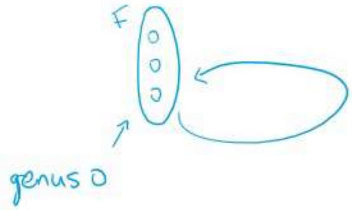




valence ~~4 ≤ 3~~

[Golla-Ghiggini-Plamenevskaya]

→ Surface singularities whose contact link is planar



link

$$F \hookrightarrow Y - B$$
$$\downarrow$$
$$S^1$$

> [Wendl] symplectic fillings of a planar contact manifold
> $\updownarrow$
> factorizations of monodromy into positive Dehn twists

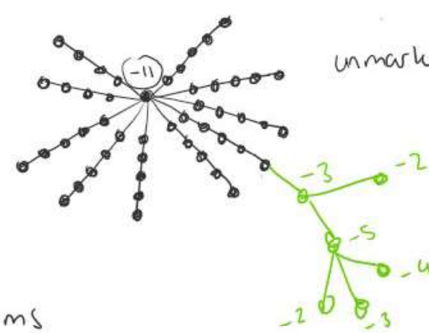
Theorem [Plamenevskaya-S.] For any $N>0$ there is an RFC singularity with at least N non-diffeomorphic (simply connected) unexpected symplectic fillings

$b_1 = 0$

different from smoothings + resolutions

not diffeomorphic rel boundary

1st case:



unmarked euler numbers = -2

higher N: more + longer arms

generalizing: any graph with such a subgraph

Contrast:

Thm [Plamenevskaya-S.] If the resolution graph for an RFC sing has all Euler numbers ≤ -5, then the only symplectic fillings are resolutions.

— Thm [Plamenevskaya-S.] If the resolution graph for an RFC sing has
Euler numbers ≤ -5, then the only symplectic fillings are resolutions.

How do we prove these results?

Complex smoothings of RFC surface sing
[de Jong-van Straten]
certain deformations of associated plane curve sing

complex deformations

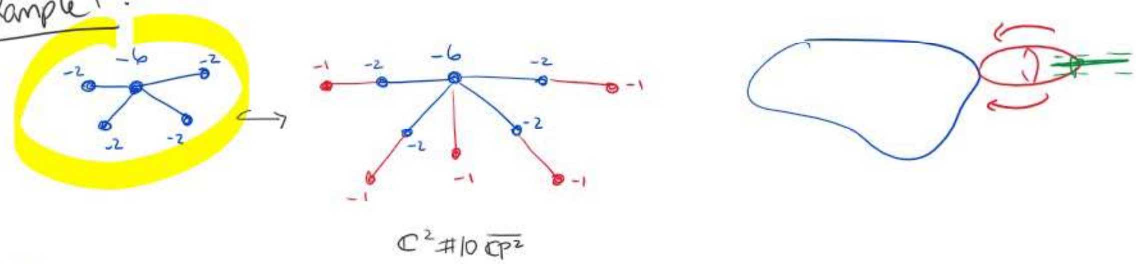
Symplectic fillings of RFC surf sing
[Plamenevskaya-S.]
"graphical homotopies" of associated plane curve sing

Smooth homotopies of plane curve components

Unexpected fillings come from curve arrangements which are related to a plane curve singularity by graphical homotopy but not complex deformation.

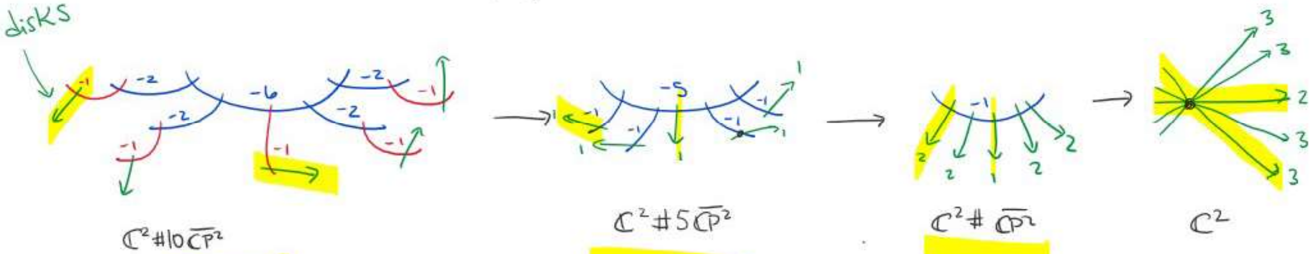
RFC singularity $\xrightarrow{\text{[de Jong-van Straten]}}$ plane curve singularity
(weighted)

Example 1:



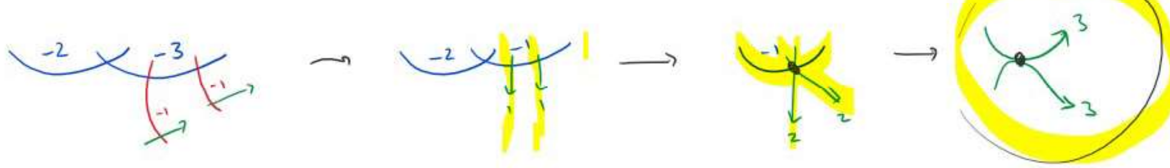
$\mathbb{C}^2 \# 10\overline{\mathbb{CP}^2}$

disks



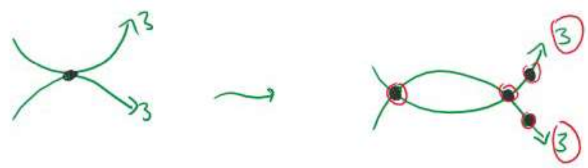
$\mathbb{C}^2 \# 10\overline{\mathbb{CP}^2}$ $\mathbb{C}^2 \# 5\overline{\mathbb{CP}^2}$ $\mathbb{C}^2 \# \overline{\mathbb{CP}^2}$ $\mathbb{C}^2$

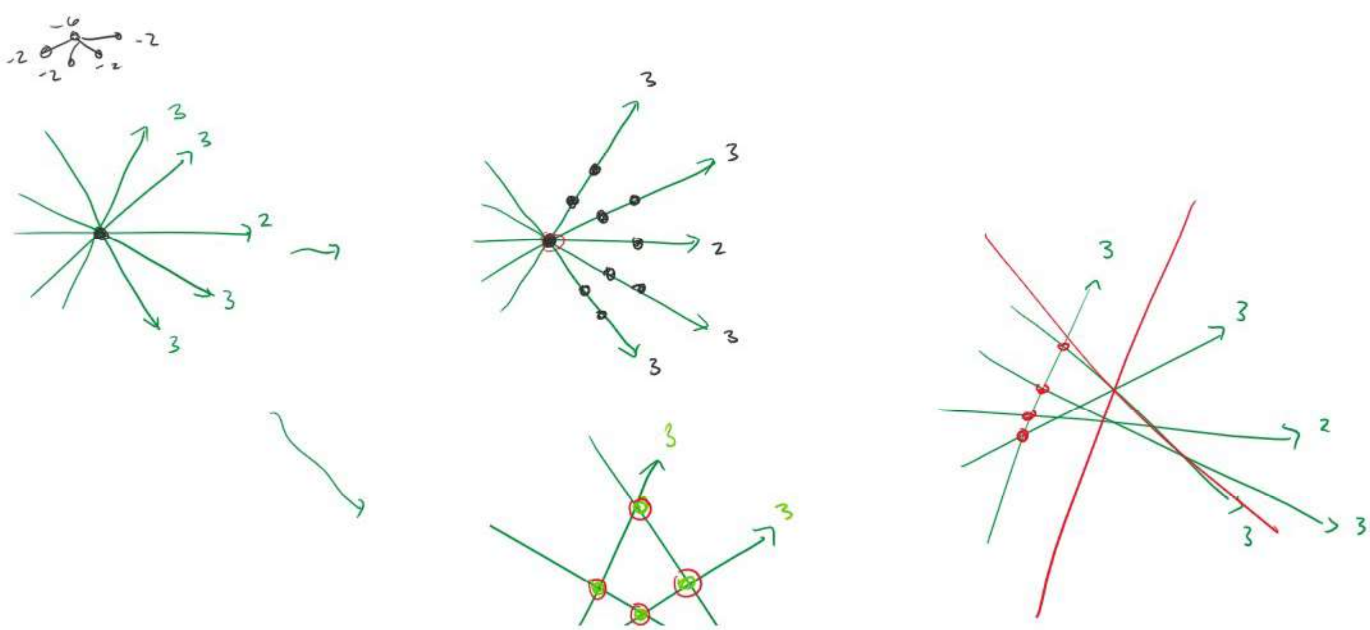
Example 2:

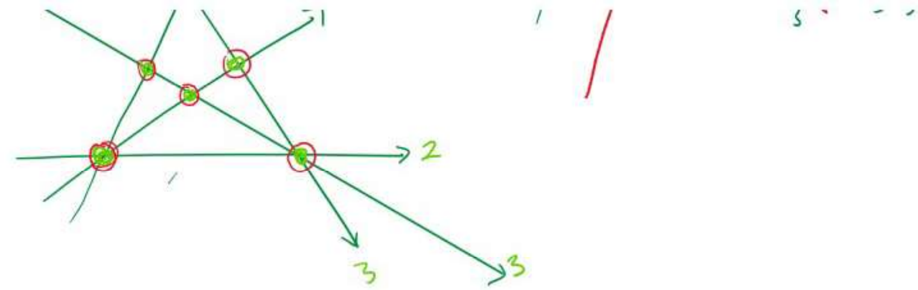


Complex smoothings $\xleftrightarrow{[dJvS]}$ Complex deformations of the sing plane curve to one with only transverse intersections
+ weight condition

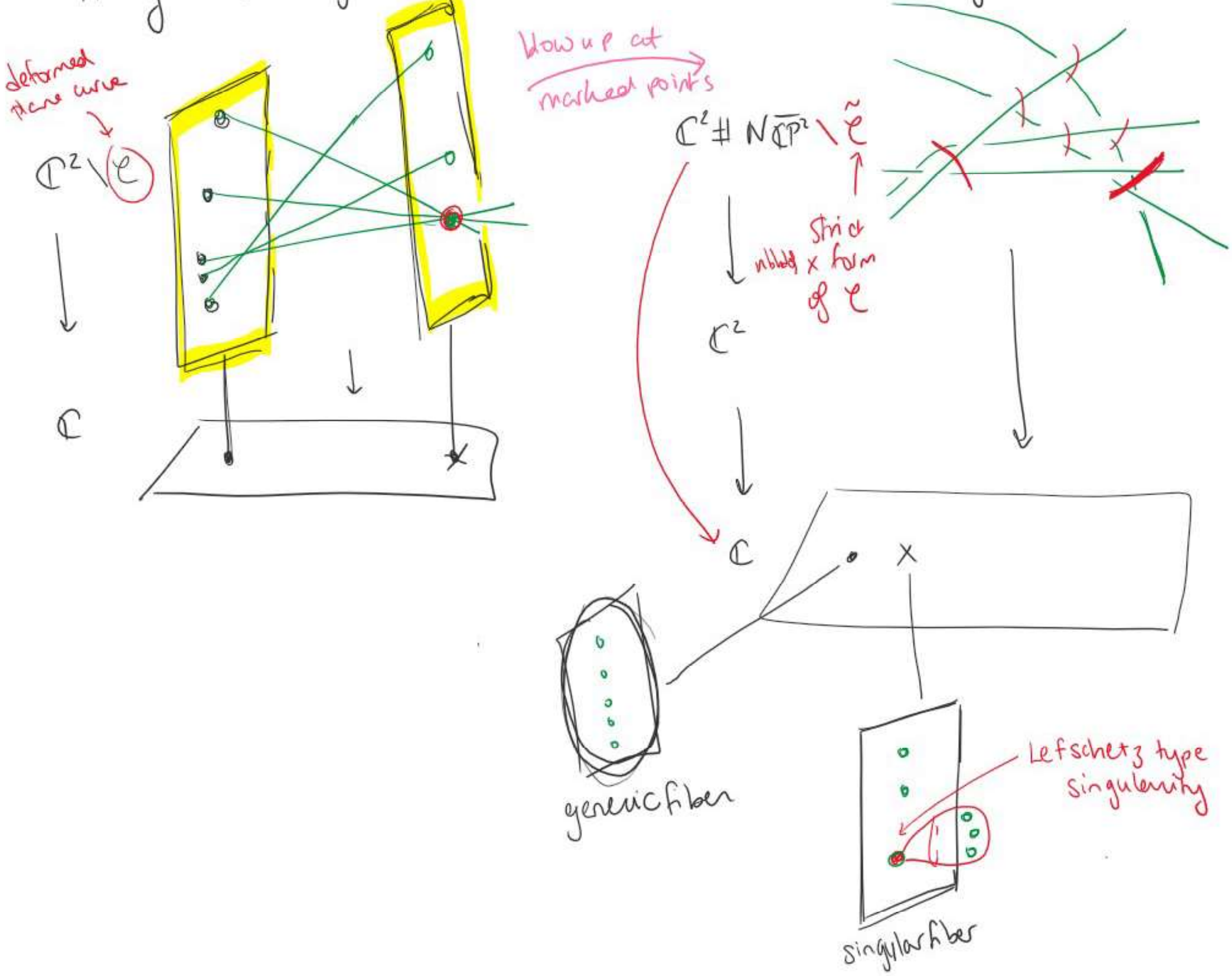
Symplectic fillings $\xleftrightarrow{[PS]}$ graphical homotopies of the sing plane curve to one with only transverse intersections
+ weight condition



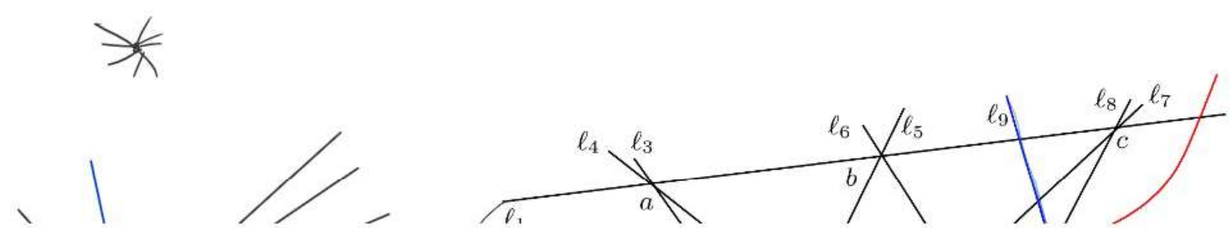


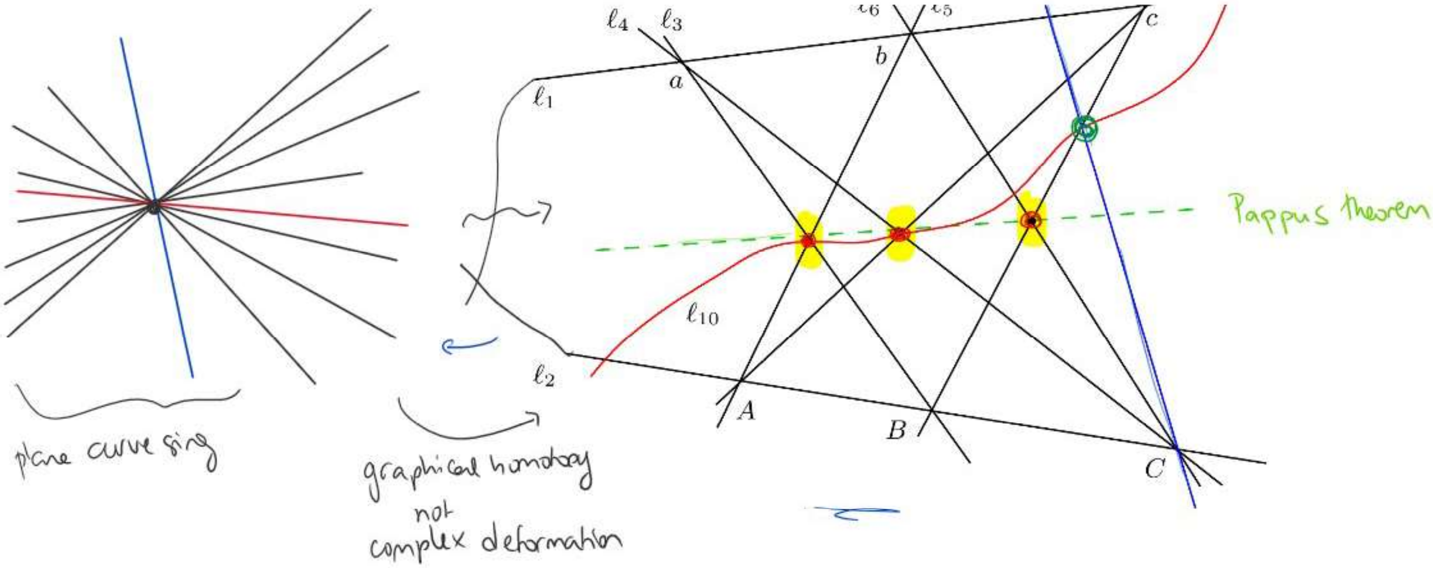


Filling arises by blowing up at marked points + removing proper xform:

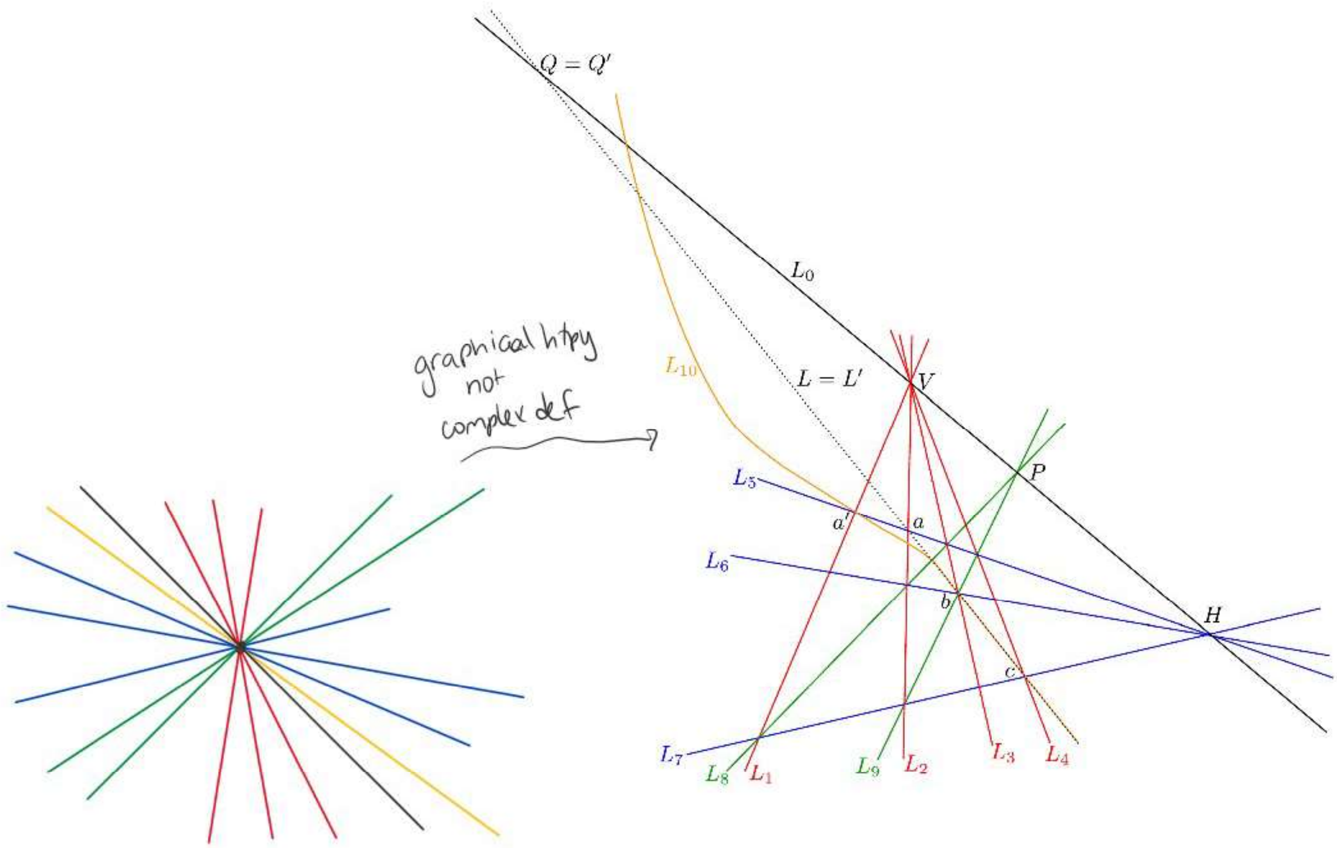


Unexpected fillings come from unexpected configurations





ℓ_4 ℓ_3 ℓ_6 ℓ_5
c
a
b
ℓ_1
Pappus theorem
ℓ_{10}
ℓ_2
A
B
C
plane curve sing
graphical homotopy not complex deformation



$Q = Q'$
L_0
L_{10}
$L = L'$
V
graphical htpy not complex def
P
L_5
a'
a
L_6
b
H
c
L_7
L_8 L_1
L_9 L_2
L_3 L_4