Caustics of Lagrangian homotopy spheres with stably trivial Gauss map

\[ \begin{align*}
\text{§1 Main result} & \quad (10) \\
\text{§2 Applications} & \quad (15) \\
\text{§3 Homotopical calculation} & \quad (15) \\
\text{§4 The exceptional dimensions } n = 3, 7 & \quad (15)
\end{align*} \]

\[ \begin{align*}
\text{TODAY:} & \quad \{ \begin{align*}
(1) & \text{ IOU from PhD thesis} \\
(2) & \text{ UROP with D. Darrow}
\end{align*} \] \\
\text{LC}(M^n, \omega) & \text{ Lagrangian htpy sphere} \\
\text{DC TM} & \text{ Lagrangian distribution.}
\end{align*} \]

**Def** \( \mathcal{L} \) is stably trivial if \( \Theta \) and \( \Theta' \) are homotopic as Lagrangian distributions in \( TM|_{\mathcal{L} \oplus C} \).

**Thm** The tangencies of \( L \) wrt \( \mathcal{L} \) can be simplified to consist only of folds via a Hamiltonian isotopy \( \leftrightarrow \) \( \mathcal{L} \) is stably trivial.
Remarks:

- A fold is the simplest tangency:
  \[ p^2 = p^1 \times \mathbb{R}^{n-1} \subset T^*\mathbb{R} \times T^*(\mathbb{R}^{n-1}) = T^*\mathbb{R}^n \]
  \[ \zeta = \ker(d\zeta), \; \pi : T^*\mathbb{R}^n \to \mathbb{R}^n \]

- \( T(T^*L) \mid_L \cong L \times C^n \) so can view htpy class of \( \gamma_1 \mid_L \) as element of \( \pi_n \wedge n \), where \( \wedge_n = \bigwedge^n \Omega_n \). Then stable triviality means

  \[ \left[ \gamma_1 \mid_L \right] \in \ker(\pi_n \wedge n \to \pi_n \wedge n+1) \]

- \( \pi_n \wedge n+1 \cong \pi_n \wedge = 0 \quad n \equiv 0, 4, 6, 7 \ (8) \)

In my PhD I proved:

Theorem: The tangencies of \( L \) wrt \( \zeta \) can be simplified to consist only of folds via a Hamiltonian isotopy \( \iff \zeta \) is homotopic to a \( \zeta' \) wrt which \( L \) only has folds.

Therefore the problem reduces to:
Thm Every $\alpha \in \ker(\pi_n \Lambda n \to \pi_{n+1} \Lambda n+1)$ admits a rep $(D^n, \partial D^n) \to (\Lambda n, i^* \mathbb{R}^n)$ with only fold tangencies over $\mathbb{R}^n$.

Remarks:
- $\ker(\pi_n \Lambda n \to \pi_{n+1} \Lambda n+1) = \begin{cases} \mathbb{Z} & n=0 \quad (2) \\ \mathbb{Z}/2 & n \geq 1 \quad (n>1) \end{cases}$

$\S 2$. (A) Let $(\omega, \lambda, \phi)$ be Weinstein s.t
(1) $\exists \mathcal{C}$ CW global field of Lag. planes
(2) $\phi$ is Morse with 2 critical points.

Corollary $(\omega, \lambda, \phi)$ is homotopic to a Weinstein domain with arboreal skeleton.

Remarks:
- Recovers special case of more general result joint with Eliashberg & Nadler without hypothesis (2).
- Difficulties in general case arise from interaction of $\geq 3$ strata.
- Here can use Starkston's strategy, used to arborealize 4D Weinstein manifolds.
Proof: $\text{Crit}(\phi) = \{ x_0, x_n \}$, $\text{ind}(x_i) = i$.

$W^s(x_n) \cap S^*D^n \to S\text{kel}$

$U^*D^n$

$\omega lg : \ker(d\pi)$ on $U^*D^n$, where $\pi: U^*D^n \to D^n$ proj. Then $\pi|_\Lambda = \nu \oplus \mathbb{Z}$ for $\mathbb{Z}$ Liouville direction. Singularities of $\pi|_\Lambda: \Lambda \to D^n \cong$ tangencies of $\Lambda$ w.r.t $\nu$.

But $\nu \oplus \mathbb{Z}$ extends to $W^s(x_n) \Rightarrow \nu$ is stably trivial. Thm applies.

Starkston

QED
(B) Let $LCT^* \Sigma$ Lagrangian htpy sphere.

Corollary $\exists \Phi_t$ Ham isotopy s.t $\Phi_t(L)$ is generated by a framed function on some tube bundle.

Remarks:
- A tube $TC_{\mathbb{R}^{N+1}}$ is, up to a compactly supported isotopy, the result of attaching a standard handle to $\partial x \mathbb{N}^{N+1} \leq 0$.

- A framed function $f: W \to \mathbb{R}$ on a fibre bundle $F \to W \to B$ is:
  1. $f_b: F_b \to \mathbb{R}$ Morse / generalized Morse $\forall b \in B$.
  2. Negative eigenspaces at critical points of $f_b$ are framed.

Framed functions are the homotopically canonical way of studying smooth fibre bundles via parametrized Morse theory.
Proof: Heavy lifting is done by:

Thm (Abouzaid, Courte, Guillermou, Kragh) \( LCT^{*}\Sigma \) admits a generating function on some tube bundle \( W \to \Sigma \).

This implies \( \Sigma \to U/\Sigma \) null homotopic, hence our theorem applies with \( x = \kappa_{\alpha}(d\alpha) \), \( \pi : T^{*}\Sigma \to \Sigma \).

By homotopy lifting property can cover \( \Phi_t(L) \) with generating functions \( f_t \) on a stabilization of \( W \).

Since \( \Phi_1(L) \Delta \kappa \) only have fold tangencies, \( f_1 \) restricts to each fibre as a Morse or generalized Morse function.

Now, \( f_1 \) might not admit a framing but this can be corrected by a twisted stabilization of \( W \) using Abouzaid: \( \Phi_1(L) \to \Sigma \) Htpy equiv. 

\( \Box \)
§3  We rely on the fibrations
\[ \varnothing^n \rightarrow U^n \rightarrow \Lambda^n \]
\[ \downarrow \quad \downarrow \quad \downarrow \]
\[ \varnothing^{n+1} \rightarrow U^{n+1} \rightarrow \Lambda^{n+1} \]
& their LES in homotopy.

**Lemma 1** For \( n \neq 1, 3, 7 \) we have
\[ \ker(\pi_n \Lambda^n \rightarrow \pi_n \Lambda^{n+1}) \cap \ker(\pi_{n-1} \Omega^n \rightarrow \pi_{n-1} \Omega^{n+1}) = 0. \]

Proof: By cases on \( n \) modulo 8, using results in the literature on the homotopy groups of the classical groups in metastable range (Bott, Kervaire, Milnor, Kochi ...)

**Definition** A formal fold \((\Sigma, v)\) in \( IR^n \) is:
1. \( \Sigma \subset IR^n \) smooth compact hypersurface
2. \( v \) co-orientation of \( \Sigma \).

Remark: special case of Entov's notion of chain.
To \((\Sigma,v)\) associate a Lagrangian dist in \(T^*\mathbb{R}^n\) with folds along \(\Sigma\) & Maslov co-orientation \(v\). This determines \(\alpha(\Sigma,v) \in \pi_n \Lambda_n\).

**Lemma 2** The images of the \(\alpha(\Sigma,v)\) under \(\pi_n \Lambda_n \to \pi_{n-1} \Omega_n\) generate the subgroup \(\ker (\pi_{n-1} \Omega_n \to \pi_n \Omega_n)\).

**Proof:**
\[
\ker (\pi_{n-1} \Omega_n \to \pi_n \Omega_n) \\
= \text{im} (\pi_n S^n \to \pi_{n-1} \Omega_n) = <TS^n> + \text{Poincaré-Hopf. QED}
\]

**Corollary:** For \(n \neq 1,3,7\) there is an iso:
\(\ker (\pi_n \Lambda_n \to \pi_n \Lambda_{n+1}) \to \ker (\pi_{n-1} \Omega_n \to \pi_{n-1} \Omega_{n+1})\).

**Proof:** Lemma 1 \(\Rightarrow\) inj
Lemma 2 \(\Rightarrow\) surj. QED

**Proof of Thm for \(n \neq 1,3,7\):**
Let \(\gamma \in \ker (\pi_n \Lambda_n \to \pi_n \Lambda_{n+1})\). By Lemma 2, \(\exists (\Sigma,v)\) s.t. \(\gamma \neq \alpha(\Sigma,v)\) are equal in \(\pi_{n-1} \Omega_n\), hence also equal in \(\pi_n \Lambda_n\). QED
§4. n=1 is trivial for n=3,7 the above doesn’t work (π_{20}O_3=π_{60}O_7=0). Alternative approach: we construct \( F: T(T^*S^n) |_{S^n} \xrightarrow{\sim} S^n \times \mathbb{C}^n \) s.t. 
(1) \( F^{-1}(i\mathbb{R}^n) \) has folds w.r.t \( S^n \).
(2) \( \hat{F}: T(T^*S^n) |_{S^n} \times \mathbb{C} \to S^n \times \mathbb{C}^{n+1} \)
\( \hat{F}: S^n \times \mathbb{C}^{n+1} \to S^n \times \mathbb{C}^{n+1} \)
is trivial in \( \pi_{n+1} \). Then:
\( (n=3) \ker(\pi_3 \Lambda_3 \to \pi_3 \Lambda_4) \to \pi_3 \Lambda_3 \to \pi_3 \Lambda \to 0 \)
\( 0 \to 2 \cdot 2/4 \to \mathbb{Z}/4 \to \mathbb{Z}/2 \to 0 \)
Quaternions yield a trivialization \( T\mathbb{S}^2 \xrightarrow{\sim} \mathbb{S}^3 \times i\mathbb{R}^2 \) which complexifies to \( G: T(T^*\mathbb{S}^3) |_{\mathbb{S}^3} \to \mathbb{S}^3 \times \mathbb{C}^3 \). Put:
\( \beta = F \circ G^{-1} \in \pi_3 \Lambda_3 \). By (2),
\( \forall \alpha \in \pi_3 \Lambda_3 \) is just \( \alpha(\mathbb{S}^2,\mathbb{N}B_3) \). So, we need to show:

**Claim:** \( \alpha \in \pi_3 \Lambda_3 \) is equal to twice a generator.
Proof: \( \pi_3 \mathbb{U}_3 \rightarrow \pi_3 \mathbb{A}_3 \rightarrow \pi_2 \mathbb{O}_3 = 0 \), so enough to show \( \hat{\beta} \in \pi_3 \mathbb{U}_3 \) is twice a generator. Now, \( \pi_3 \mathbb{U}_3 \sim \pi_3 \mathbb{U}_4 \), so enough to show \( \hat{\beta} \in \pi_3 \mathbb{U}_4 \) is twice a generator. But:

\[ \hat{\beta} = \hat{F} \circ \hat{G}^{-1} = \hat{G}^{-1} \in \pi_3 \mathbb{U}_4 \]

is the complexification of quaternion mult, which is \( \eta \in \pi_3 \mathbb{O}_4 \).

**Known:** \( \eta \) maps to a generator in \( \pi_3 \mathbb{O} \).

**Consider:**

\[ \pi_3 \mathbb{O}_4 \rightarrow \pi_3 \mathbb{U}_4 \]

\[ \pi_2 \mathbb{O}_5 \rightarrow \pi_3 \mathbb{U}_5 \rightarrow \pi_3 \mathbb{A}_5 \rightarrow \pi_2 \mathbb{O}_5 \]

\[ \mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}/2 \rightarrow 0 \]

It follows that the image \( \hat{\beta} \) of \( \eta \) in \( \pi_3 \mathbb{U}_4 \) is twice a generator, as claimed.

\( \square \)

Similar for octonions.