Classical and New Plumbings Bounding Contractible Manifolds and Homology Balls

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Main Problem

Main Objects

We study 3- and -4-manifolds having *simple* topologies.

Definition

- A closed, connected, oriented 3-manifold Y is called a **homology 3-sphere** if $H_*(Y, \mathbb{Z}) = H_*(S^3, \mathbb{Z})$.
- A compact, connected, oriented 4-manifold W is called a **homology 4-ball** if $H_*(W, \mathbb{Z}) = H_*(B^4, \mathbb{Z})$.
- A compact, connected, oriented 4-manifold W is called a **contractible 4-manifold** if the identity map of W is null-homotopic; equivalently, if W is a homology 4-ball with $\pi_1(W) = 0$.

Notation

- *homology 3-sphere* $\rightsquigarrow \mathbb{Z}HS^3$.
- *homology 4-ball* $\rightsquigarrow \mathbb{Z}HB^4$.

Main Problem

The analogue of the interaction $S^3 = \partial B^4$ creates our main problem:

Main Problem (Problem 4.2, Kirby's list)

Which $\mathbb{Z}HS^3$'s bound contractible 4-manifolds or $\mathbb{Z}HB^4$'s?

Freedman completely resolved this problem in the topological category.

Theorem (Freedman, 1982)

Every $\mathbb{Z}HS^3$ bounds a topological contractible 4-manifold.

Thus, we impose an extra smoothness condition.

Main Problem

Main Problem (Updated, Problem 4.2, Kirby's list)

Which $\mathbb{Z}HS^3$'s bound smooth contractible 4-manifolds or smooth $\mathbb{Z}HB^4$'s?

In the smooth case, the question is more subtle.

Answer (Positive)

Some $\mathbb{Z}HS^3$'s do bound such 4-manifolds.

$\Uparrow$

Method (Constructive)

Do Kirby calculus.

Answer (Negative)

Some $\mathbb{Z}HS^3$'s do not such 4-manifolds.

$\Uparrow$

Method (Obstructive)

Compute invariants.

Motivation:

Homology Cobordism Group

Homology Cobordism

Definition

The **homology cobordism group** $\Theta^3_{\mathbb{Z}}$ is defined as

$$\Theta^3_{\mathbb{Z}} = \{\mathbb{Z}HS^3\,'s\}/\sim$$

where the equivalence relation *homology cobordism* $\sim$ is given by

$$Y_0 \sim Y_1 \iff \partial W = -(Y_0)\#Y_1 \text{ for some smooth } \mathbb{Z}HB^4\ W.$$

Fact

A $\mathbb{Z}HS^3$ bounds a $\mathbb{Z}HB^4$ if and only if it is homology cobordant to S^3.

Structure of $\Theta^3_{\mathbb{Z}}$

Theorem (Dai-Hom-Stoffregen-Truong, 2018)

$\Theta^3_{\mathbb{Z}}$ has a $\mathbb{Z}^\infty$ summand.

Problem (Open Questions)

Is $\Theta^3_{\mathbb{Z}}$ in fact $\mathbb{Z}^\infty$? Does $\Theta^3_{\mathbb{Z}}$ contain any torsion $\mathbb{Z}_n$ for $n \geq 2$?

We may ask that what type of manifolds can(not) generate $\Theta^3_{\mathbb{Z}}$?

Theorem (Livingston, 1981; Myers, 1983; Mukherjee, 2020; Hendricks-Hom-Stoffregen-Zemke, 2020)

- *$\Theta^3_{\mathbb{Z}}$ is generated by irreducible $\mathbb{Z}HS^3$'s,*
- *$\Theta^3_{\mathbb{Z}}$ is generated by hyperbolic $\mathbb{Z}HS^3$'s,*
- *$\Theta^3_{\mathbb{Z}}$ is generated by Stein fillable $\mathbb{Z}HS^3$'s.*
- *$\Theta^3_{\mathbb{Z}}$ is not generated by Seifert fibered $\mathbb{Z}HS^3$'s.*

Plumbed Manifolds

and

Mazur's Argument

Examples of Plumbed Homology 3-Spheres

We study $\mathbb{Z}HS^3$'s which appear as the boundaries of plumbed 4-manifolds which can be obtained by plumbing 2-disk bundles over 2-sphere.

1. Seifert fibered spheres $M(S^2; a_1, \ldots, a_n)$ with n-*fibers*: given coprime positive integers $a_1, \ldots, a_n$, they are $\mathbb{Z}HS^3$'s which admit a fixed point free action of S^1 over S^2.

2. Brieskorn spheres $\Sigma(p, q, r)$: given coprime positive integers p, q and r, they are $\mathbb{Z}HS^3$'s defined as the link of the singularity at the origin

$$\Sigma(p, q, r) = \{(x, y, z) \in \mathbb{C}^3 \ : \ x^p + y^q + z^r = 0\} \cap S^5_\epsilon$$

where S^5_ϵ is 5-dimensional sphere with arbitrarily small radius ϵ.

There is a diffeomorphism: $M(S^2; a_1, a_2, a_3) \approx \Sigma(a_1, a_2, a_3)$.

Mazur Manifolds

Argument (Mazur, 1961)

Attach a 4-dimensional 2-handle $B^2 \times B^2$ to $S^1 \times B^3$ along a knot $J \subset S^1 \times S^2 = \partial(S^1 \times B^3)$:

$$W \doteq S^1 \times B^3 \bigcup_{J \subset S^1 \times S^2} B^2 \times B^2.$$

Then W is a contractible 4-manifold with one 0-handle, one 1-handle and one 2-handle because

- *J generates $\pi_1(S^1 \times B^3)$ so that W is simply-connected,*
- *W is a $\mathbb{Z}HB^4$.*

Definition

Such a 4-manifold W is so-called a **Mazur manifold**.

Mazur Manifolds

Observation

- $S^1 \times S^2$ *is* 0*-surgery on the unknot* U*, i.e.,* $S^1 \times S^2 = S^3_0(U)$.
- *The unknot* U *bounds a smoothly properly embedded disk* $D \subset B^4$. *Also* $\nu(D) \approx D \times B^2$.
- $S^1 \times S^2 = \partial(S^1 \times B^3) = \partial(B^4 - \nu(D))$.

Theorem (Mazur, 1961)

Any $\mathbb{Z}HS^3$ *obtained by an integral surgery on a knot in* $S^1 \times S^2$ *bounds a Mazur manifold with one* 0*-handle, one* 1*-handle, and one* 2*-handle.*

Slice and Ribbon Knots

Definition

A knot $K \subset S^3$ is said to be **slice knot** if K bounds a smooth properly embedded disk $D \subset B^4$. Here, D is called **slice disk**.

Definition

If $K \subset S^3$ bounds a slice disk $D \subset B^4$ with no 2-handles, then K is called a **ribbon knot**, and such a disk D is called a **ribbon disk**.

Equivalently, knot K in S^3 is a ribbon knot if it can be built by attaching bands to a several component unlink.

Definition (Miyazaki, 1986)

The **fusion number** of a ribbon knot K is the minimal number of bands to produce a ribbon disk for K.

Ribbon Knots

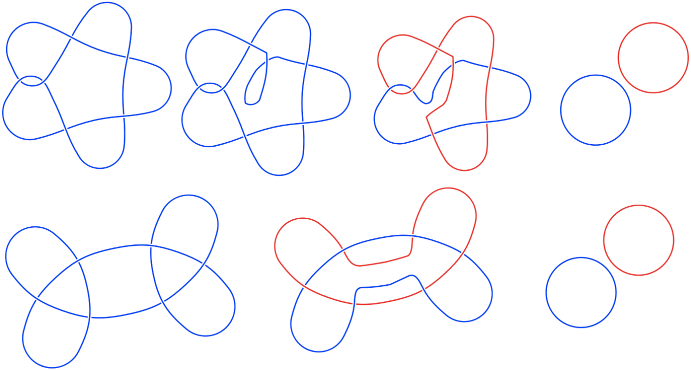


Figure: Stevedore and square knot have fusion number 1
[**Credit:** Knot-Like Objects (KLO)]

Modification of Mazur's Argument

Let K be a ribbon knot bounding ribbon disk D with fusion number n and let $Y = S^3_0(K)$. Then

- $\partial(B^4 - \nu(D)) = Y$, and
- the ribbon disk exterior $B^4 - \nu(D)$ has a single 0-handle, $n+1$ 1-handles and n 2-handles.

Lemma (Şavk, 2020)

Any $\mathbb{Z}HS^3$ obtained by an integral surgery on a knot in Y bounds a contractible 4-manifold with one 0-handle, $n+1$ 1-handles and $n+1$ 2-handles.

Definition

We call such manifolds **generalized Mazur manifolds**.

Homology Spheres

Bounding

Contractible Manifolds

Brieskorn Spheres, Contractible Manifolds and $\mathbb{Z}HB^4$'s

The classical results around 1980's indicate that

Theorem (Akbulut-Kirby, 1979; Casson-Harer, 1981; Stern 1979; Fintushel-Stern, 1981; Fickle, 1984)

The following Brieskorn spheres bound Mazur manifolds with one 0-handle, one 1-handle and one 2-handle:

- $\Sigma(2,5,7)$, $\Sigma(3,4,5)$, $\Sigma(2,3,13)$,
- $\Sigma(p, ps-1, ps+1)$ *for p even and s odd,*
- $\Sigma(p, ps\pm 1, ps\pm 2)$ *for p odd,*
- $\Sigma(2, 2s\pm 1, 2.2(2s\pm 1)+2s\mp 1)$ *for s odd,*
- $\Sigma(3, 3s\pm 1, 2.3(3s\pm 1)+3s\mp 2)$,
- $\Sigma(3, 3s\pm 2, 2.3(3s\pm 2)+3s\mp 1)$,
- $\Sigma(2,3,25), \Sigma(2,7,19)$, $\Sigma(3,5,19)$.

In addition, $\Sigma(2,7,47)$ and $\Sigma(3,5,49)$ bound $\mathbb{Z}HB^4$'s.

Classical $\mathbb{Z}HS^3$'s Bounding Generalized Mazur Manifolds

We first observe that examples of Akbulut-Kirby, Fickle and Stern also bound generalized Mazur manifolds.

Theorem (Ş., 2020)

The following Brieskorn spheres bound generalized Mazur manifolds with one 0-handle, two 1-handles, and two 2-handles:

- $\Sigma(2,3,13)$ *and* $\Sigma(2,3,25)$,
- $\Sigma(2,4n+1,20n+7)$,
- $\Sigma(3,3n+1,21n+8)$,
- $\Sigma(2,4n+3,20n+13)$,
- $\Sigma(3,3n+2,21n+13)$.

Maruyama Manifolds Bounding Mazur Manifolds

We have also non Seifert fibered plumbed homology spheres:

Theorem (Maruyama, 1982; Akbulut-Karakurt, 2014)

Let $X(n)$ be Maruyama plumbed 4-manifold in the figure. Then for each $n \geq 1$ its boundary $\partial X(n)$ is a $\mathbb{Z}HS^3$ which bounds a Mazur manifold with one 0-handle, one 1-handle and one 2-handle.

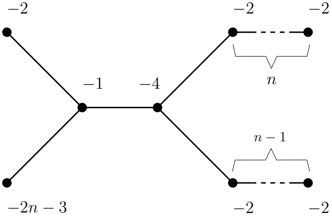


Figure: Maruyama plumbed 4-manifold $X(n)$.

New Manifolds Bounding Mazur Manifolds

Recently, we found a new family of plumbed homology spheres:

Theorem (Ş., 2020)

Let $X'(n)$ be the companion of Maruyama plumbed 4-manifold in the following figure. Then for each $n \geq 1$ its boundary $\partial X'(n)$ is a $\mathbb{Z}HS^3$ which bounds a Mazur manifold with one 0-handle, one 1-handle and one 2-handle.

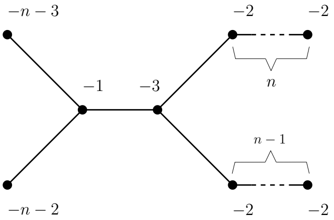


Figure: The companion of Maruyama plumbed 4-manifold $X'(n)$.

New Manifolds Bounding New Mazur Manifolds

We exhibit one more new infinite family:

Theorem (Ş., 2020)

Let $W(n)$ be Ramanujam plumbed 4-manifold in the following figure. Then for each $n \geq 1$ its boundary $\partial W(n)$ is a $\mathbb{Z}HS^3$ which bounds a generalized Mazur manifold with one 0-handle, two 1-handles, and two 2-handles.

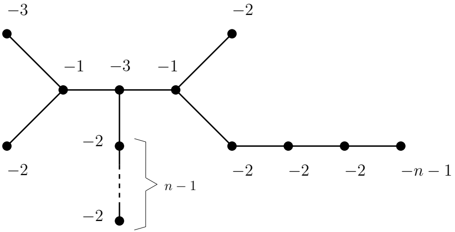
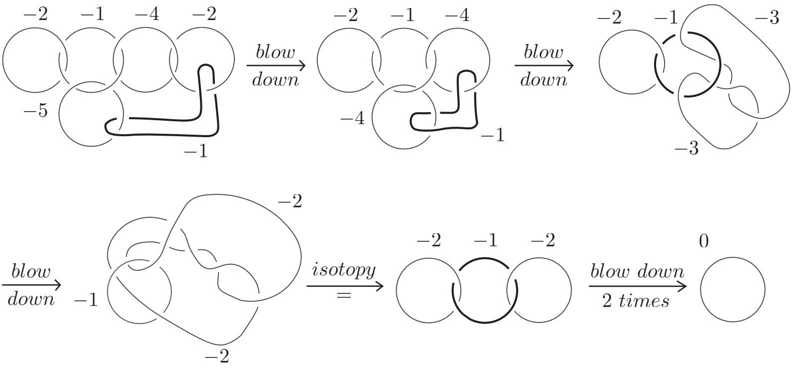


Figure: Ramanujam plumbed 4-manifold $W(n)$.

Recovery: Brieskorn Sphere $\Sigma(2,5,7)$

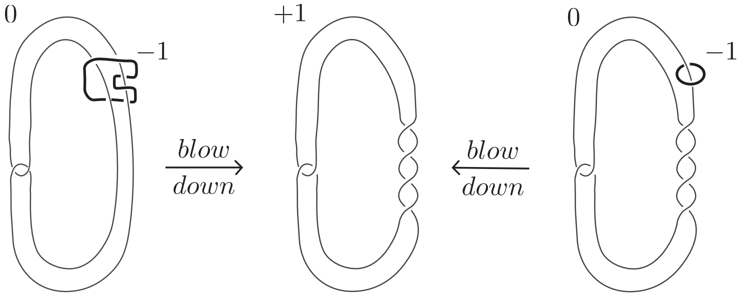
Our approach also provides simple proofs for the classical results:

The Brieskorn sphere $\Sigma(2,5,7)$ bounds a Mazur manifold:



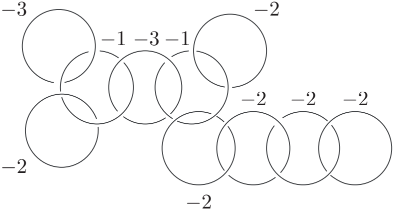
Recovery + New: Brieskorn Sphere $\Sigma(2,3,13)$

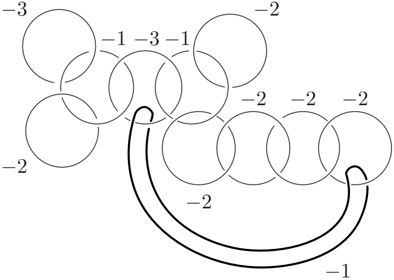
The Brieskorn sphere $\Sigma(2,3,13)$ can be obtained by $(+1)$-surgery on the stevedore knot. Thus, it bounds a Mazur manifold and a generalized Mazur manifold:



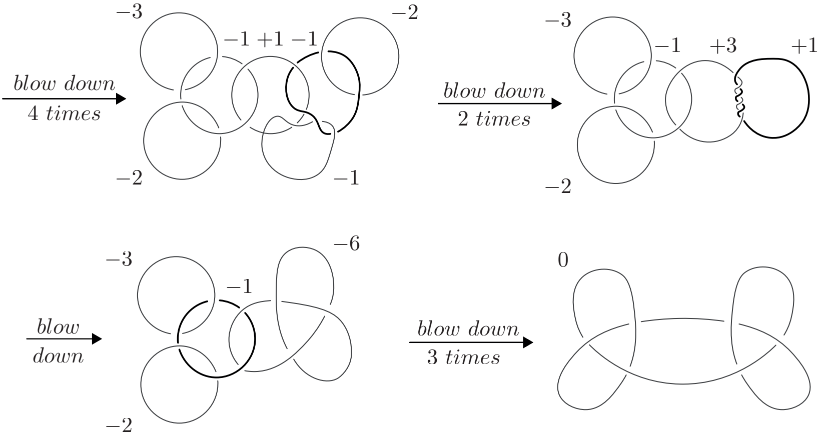
New Example: Ramanujam manifold

The boundary $\partial W(1)$ bounds a generalized Mazur manifold:





New Example: Ramanujam manifold



Mazur Manifolds are Stein

Theorem (Eliashberg, 1990)

Let X be a smooth 4-manifold. If X has the handle decomposition with

- 0-*handle(s),*
- 1-*handle(s), and*
- 2-*handle(s),*

then X admits a Stein structure.

Corollary

The Mazur and generalized Mazur manifolds are both Stein.

Homology Spheres

Bounding

Rational Homology Balls

Brieskorn Spheres Bounding $\mathbb{Q}HB^4$'s

Bounding $\mathbb{Z}HB^4$'s a priori implies bounding $\mathbb{Q}HB^4$'s.

Definition

A $\mathbb{Z}HS^3$ is said to be **non-trivially** bounds a $\mathbb{Q}HB^4$ if it is obstructed from bounding a $\mathbb{Z}HB^4$.

Considering the rational version of $\Theta^3_{\mathbb{Z}}$, we ask:

Question

Which $\mathbb{Z}HS^3$'s non-trivially bound $\mathbb{Q}HB^4$'s? Equivalently, can we find non-trivial elements in $\mathrm{Ker}(\Theta^3_{\mathbb{Z}} \to \Theta^3_{\mathbb{Q}})$?

Fintushel and Stern provided the first example.

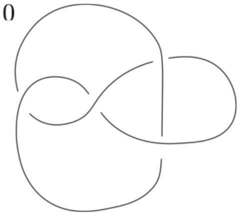
Theorem (Fintushel-Stern, 1984)

The Brieskorn sphere $\Sigma(2,3,7)$ non-trivially bounds a $\mathbb{Q}HB^4$. Therefore, $\mathrm{Ker}(\Theta^3_{\mathbb{Z}} \to \Theta^3_{\mathbb{Q}})$ contains a $\mathbb{Z}$ subgroup.

Technique of Akbulut and Larson

The Brieskorn sphere $\Sigma(2,3,7)$ has remained the single example for more than thirty years. The recent progress is started by the work of Akbulut and Larson.

Let Z denote 3-manifold obtained by 0-surgery on figure-eight knot in S^3. In handle notation,



Lemma (Akbulut-Larson, 2018)

Any $\mathbb{Z}HS^3$ obtained by an integral surgery on a knot in Z bounds a $\mathbb{Q}HB^4$.

Brieskorn Spheres Bounding $\mathbb{Q}HB^4$'s

Akbulut and Larson presented the first additional examples.

Theorem (Akbulut-Larson, 2018)

The following Brieskorn spheres non-trivially bound $\mathbb{Q}HB^4$'s:

- $\Sigma(2, 3, 19)$,
- $\Sigma(2, 4n+1, 12n+5)$ *for odd* n,
- $\Sigma(3, 3n+1, 12n+5)$ *for odd* n.

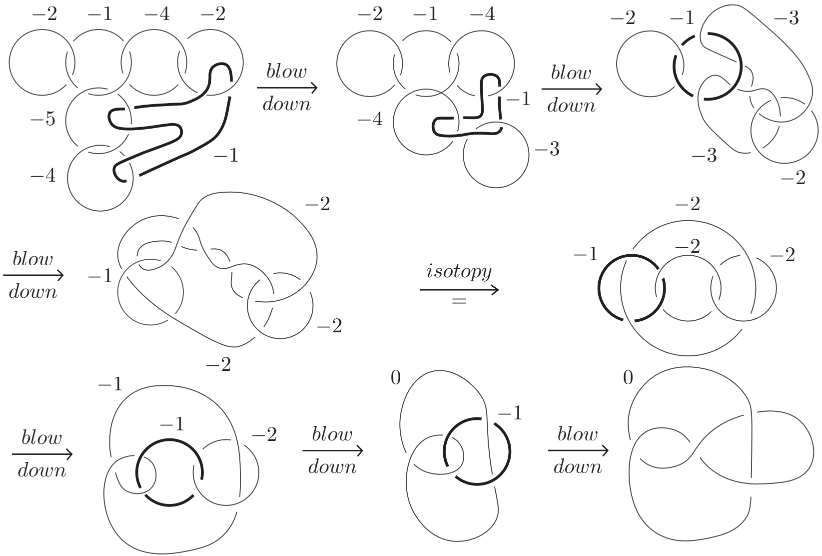
Using their technique, we found new infinite families.

Theorem (Ş., 2019)

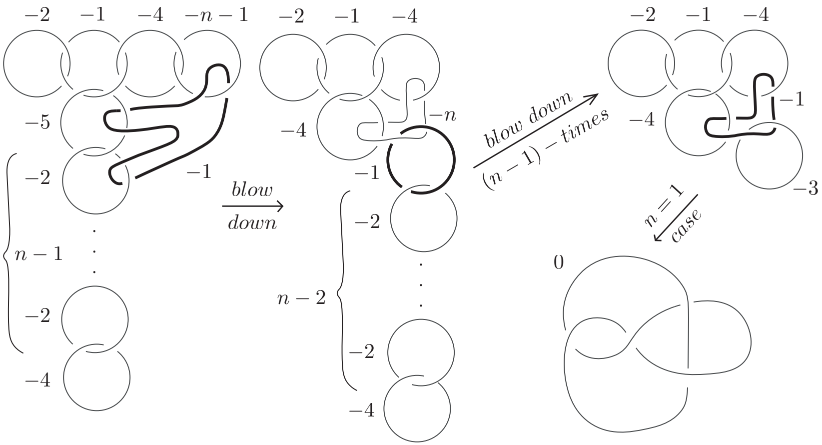
The following Brieskorn spheres also non-trivially bound $\mathbb{Q}HB^4$'s:

- $\Sigma(2, 4n+3, 12n+7)$ *for even* n,
- $\Sigma(3, 3n+2, 12n+7)$ *for even* n.

Proof of New Brieskorn Spheres: Base case



Proof of New Brieskorn Spheres: General case



Obstruction and Further Directions

The obstruction comes from the Neumann-Siebenmann invariant. Our spheres do not bound integral homology balls for even n's:

$$\bar{\mu} = \begin{cases} 1, & \text{if } n \text{ is even}, \\ 0, & \text{if } n \text{ is odd}. \end{cases}$$

All current invariants cannot detect the linear independence of these Brieskorn spheres in $\Theta^3_{\mathbb{Z}}$.

Problem (Open Question)

Does $\mathrm{Ker}(\Theta^3_{\mathbb{Z}} \to \Theta^3_{\mathbb{Q}})$ *contain a* $\mathbb{Z}^\infty$ *subgroup or a* $\mathbb{Z}^\infty$ *summand?*

Problem (Open Question)

Are they homology cobordant to each other or to S^3 *in* $\Theta^3_{\mathbb{Z}}$*?*

An Obstruction due to 3-Handles

Theorem (Eliashberg, 1990; Gompf, 1998)

A smooth 4-manifold is Stein if and only if it has a handle decomposition with

- 0-*handle(s),*
- 1-*handle(s), and*
- 2-*handles;*

and the 2-handles are attached along Legendrian knots with framing $\mathrm{tb} - 1$.

Fact

If a $\mathbb{Z}HS^3$ non-trivially bounds a $\mathbb{Q}HB^4$ X, then any handle decomposition of X must contain 3-handles.

Several Rational Balls are not Stein

Corollary

Let X be a $\mathbb{Q}HB^4$ with the boundary one of the following list:

- $\Sigma(2,3,7)$,
- $\Sigma(2,3,19)$,
- $\Sigma(2,4n+1,12n+5)$ *for odd* n,
- $\Sigma(3,3n+1,12n+5)$ *for odd* n,
- $\Sigma(2,4n+3,12n+7)$ *for even* n,
- $\Sigma(3,3n+2,12n+7)$ *for even* n.

Then X cannot admit a Stein structure.

THANKS FOR YOUR

ATTENTION!