Periodic delay orbits and the polyfold IFT

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Periodic delay orbits and the polyfold IFT

- existence result for a family of periodic delay orbits
- first example for the use of polyfold theory in the field of differential delay equations

Plan for this talk:

- Setting and main result
- Strategy
- Polyfold theory
- Some words about the proof
- Generalizations

Setting

$X : S^1 \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ smooth, time-dependent vector field on $\mathbb{R}^n$
(1-periodic in time)

$\tau \in \mathbb{R}$ parameter, called "delay"

Consider the following equation:

$$\dot{x}(t) = X_t\big(x(t-\tau)\big)$$

We look for 1-periodic solutions $x : S^1 \longrightarrow \mathbb{R}^n$.

Consider the following equation:

$$\dot{x}(t) = X_t\big(x(t-\tau)\big)$$

Remarks:

- This equation only makes sense in $\mathbb{R}^n$. On a manifold, one may consider more complicated delay equations.
- The equation is non-local: $\dot{x}(t)$ depends on the "past" of the trajectory x. Thus, everything is more complicated than for ordinary differential equations. Solutions do not form a flow on $\mathbb{R}^n$, rather a semi-flow on a function space.
- The equation is not smooth in τ.

For $\tau = 0$ we recover orbits of the vector field X.

We vary the delay τ and study 1-periodic solutions.

Theorem (Albers–S.)

Assume that x_0 is a non-degenerate 1-periodic orbit of the vector field X.

Then for every small enough $\tau \in \mathbb{R}$ there exists a (locally unique) solution $x_\tau \in \mathcal{C}^\infty(S^1, \mathbb{R}^n)$ of the delay equation

$$\dot{x}(t) = X_t\big(x(t - \tau)\big).$$

The parametrization $\tau \longmapsto x_\tau$ is smooth.

We generalize this to

- finitely many discrete delays
- suitable delay equations on manifolds
- certain cases of time-dependent delay (work in progress).

Strategy

Solutions are zeros of the map

$$s : \mathbb{R} \times \mathcal{C}^\infty(S^1, \mathbb{R}^n) \longrightarrow \mathcal{C}^\infty(S^1, \mathbb{R}^n)$$
$$(\tau, x) \longmapsto \dot{x} - X\big(x(\cdot - \tau)\big).$$

We start with $s(0, x_0) = 0$ and claim that the zero set near $(0, x_0)$ carries the structure of a 1-dimensional smooth manifold (and is not contained in $\{0\} \times \mathcal{C}^\infty(S^1, \mathbb{R}^n)$).

$\Longrightarrow$ want to use an implicit function theorem (IFT)

Problem: The shift map $(\tau, x) \longmapsto x(\cdot - \tau)$ is not smooth.

$\Longrightarrow$ use sc-calculus and the M-polyfold IFT instead

Classical differentiability

From now on denote $H_m := W^{m,2}(S^1, \mathbb{R}^n)$, in particular $H_0 = L^2(S^1, \mathbb{R}^n)$. This gives a sequence of compact and dense embeddings

$$\cdots \hookrightarrow H_{m+1} \hookrightarrow H_m \hookrightarrow \cdots \hookrightarrow H_1 \hookrightarrow H_0.$$

For $\tau \in \mathbb{R}$ and $x \in H_0$ we define the shift of x by τ as follows:

$$\varphi(\tau, x) := x(\cdot - \tau) \in H_0.$$

Note: If $x \in H_m$, then $\varphi(\tau, x) \in H_m$ with $\|\varphi(\tau, x)\|_{H_m} = \|x\|_{H_m}$.

Fact

- *The shift map*

$$\begin{aligned} \varphi : \mathbb{R} &\longrightarrow \mathcal{L}(H_0, H_0) \\ \tau &\longmapsto \big(x \mapsto x(\cdot - \tau)\big) \end{aligned}$$

  is not continuous when the target space carries the operator topology.

- *The shift map*

$$\varphi : \mathbb{R} \times H_1 \longrightarrow H_0$$

  is continuously differentiable with

$$\mathrm{d}\varphi(\tau, x)(T, \hat{x}) = \varphi(\tau, \hat{x}) - T \cdot \varphi(\tau, \dot{x}).$$

The solution space is cut out by the map

$$\begin{aligned} s : \mathbb{R} \times H_1 &\longrightarrow H_0 \\ (\tau, x) &\longmapsto \dot{x} - X\big(\varphi(\tau, x)\big). \end{aligned}$$

From the properties of the shift map φ:

- s is continuously differentiable.
- It is not smooth on any fixed level $\mathbb{R} \times H_m$.

This means:

- We can use the $\mathcal{C}^1$-version of the IFT for Banach spaces and get a $\mathcal{C}^1$-manifold of smooth solutions $(x_\tau)_\tau$.
- This can only provide a $\mathcal{C}^1$-parametrization $\tau \longmapsto x_\tau$, not smooth.
- If as domain we take $\mathbb{R} \times H_2$, we gain another derivative of s, but we loose the Fredholm property.

But: The setting of sc-calculus comes naturally and solves the problem!

Reminder: Polyfold theory

Polyfold theory

- was developed by H. Hofer, K. Wysocki and E. Zehnder
- is used for the study of moduli spaces of J-holomorphic curves
- was made to overcome the following problems:
    - varying domains
    - varying automorphism groups
    - transversality issues
    - **non-smoothness of reparametrization maps**

All details can be found in the book “Polyfold and Fredholm theory” by HWZ which is available on Arxiv.

Main ingredients of polyfold theory:

- **sc-Banach spaces** $\mathbb{E} = (E_0 \supseteq E_1 \supseteq E_2 \supseteq \dots)$
- **sc-differentiability**: new sense of differentiability for maps between sc-Banach spaces

} **sc-calculus**

- **M-polyfolds**: topological spaces that locally look like open subsets of sc-retracts in these sc-Banach spaces, with sc-smooth transition maps
- **polyfolds**: additionally allowing for orbifold behaviour

Important results:

- chain rule for sc-differentiability
- an implicit function theorem (IFT) for sc-smooth sc-Fredholm sections

Applying the M-polyfold IFT

Theorem (IFT for M-polyfold bundles)

Assume that

- *we have a tame strong M-polyfold bundle admitting sc-smooth bump functions,*
- *f is a sc-smooth section,*
- *f has the sc-Fredholm property and*
- *f is in good position.*

Then the zero set of f is a smooth manifold of dimension equal to the Fredholm index of f.

sc-spaces:

$$\mathsf{H} = \big(H_m = W^{m,2}(S^1, \mathbb{R}^n)\big)_{m \geq 0}$$
$$\mathsf{H}^1 = \big(H^1_m = W^{m+1,2}(S^1, \mathbb{R}^n)\big)_{m \geq 0}$$
$\mathbb{R}$ with constant sc-structure

trivial strong bundle:

$$\mathbb{R} \times \mathsf{H}^1 \triangleright \mathsf{H} \longrightarrow \mathbb{R} \times \mathsf{H}^1$$

section:

$$\begin{aligned} s : \mathbb{R} \times \mathsf{H}^1 &\longrightarrow \mathsf{H} \\ (\tau, x) &\longmapsto \dot{x} - X\big(\varphi(x, \tau)\big) \end{aligned}$$

Theorem (IFT for M-polyfold bundles)

Assume that

- *we have a tame strong M-polyfold bundle admitting sc-smooth bump functions,*
  our setting: the trivial sc-Hilbert space bundle
  $$\mathbb{R} \times \mathsf{H}^1 \triangleright \mathsf{H} \longrightarrow \mathbb{R} \times \mathsf{H}^1$$
- *f is a sc-smooth section,*
  our setting: $s : \mathbb{R} \times \mathsf{H}^1 \longrightarrow \mathsf{H},\ s(\tau, x) = \dot{x} - X\big(\varphi(x, \tau)\big)$
- *f has the sc-Fredholm property and*
  some work!
- *f is in good position.*
  need transversality

Then the zero set of f is a smooth manifold of dimension equal to the Fredholm index of f.

Indeed, we could prove the following:

Theorem (Albers–S.)

1. *$s : \mathbb{R} \times \mathrm{H}^1 \longrightarrow \mathrm{H}$ is a sc-Fredholm section of index 1.*
2. *If x_0 is a non-degenerate periodic orbit of X, then $\mathrm{d}s(0, x_0) : \mathbb{R} \times H_1 \longrightarrow H_0$ is surjective.*

About the proof:

1. sc-smoothness of s is immediate from sc-smoothness of the shift map φ (which was shown by Frauenfelder–Weber). The sc-Fredholm property requires some work; we use the criteria given by Katrin Wehrheim.
2. More precisely, we show that $\mathrm{d}s(0, x_0)(0, \cdot) : H_1 \longrightarrow H_0$ is surjective if and only if x_0 is non-degenerate. This also implies that the 1-dimensional local solution set inside $\mathbb{R} \times \mathrm{H}^1$ projects to an interval around $0 \in \mathbb{R}$.

This together with the M-polyfold IFT implies our main theorem:

Theorem (Albers–S.)

Assume that x_0 is a non-degenerate 1-periodic orbit of the vector field X.

Then for every small enough $\tau \in \mathbb{R}$ there exists a (locally unique) solution $x_\tau \in \mathcal{C}^\infty(S^1, \mathbb{R}^n)$ of the delay equation

$$\dot{x}(t) = X_t\big(x(t-\tau)\big).$$

The parametrization $\tau \longmapsto x_\tau$ is smooth.

Generalizations

1) finitely many discrete delays

- equations of the form $\dot{x}(t) = X_t\big(x(t-\tau_1), \ldots, x(t-\tau_k)\big)$ with k delays $\tau_1, \ldots, \tau_k \in \mathbb{R}$ and $X : S^1 \times (\mathbb{R}^n)^k \to \mathbb{R}^n$
- by diagonal embedding, such an X defines a vector field on $\mathbb{R}^n$, and we can start with a non-degenerate orbit x_0 of this vector field
- get a k-dimensional solution manifold

2) suitable delay equations on manifolds

- e.g. equations of the form $\dot{x}(t) = f_t\big(x(t-\tau)\big)X_t\big(x(t)\big)$
- instead of linear sc-Hilbert spaces we get a sc-Hilbert manifold and a non-trivial bundle
- to insure that non-degeneracy of x_0 implies surjectivity we need to assume that $x_0^* TM$ is trivial
- get a 1-dimensional solution manifold as before

3) certain cases of time-dependent delay in $\mathbb{R}^n$ (work in progress)

- equations of the form $\dot{x}(t) = X_t\big(x(t - \tau(t))\big)$
  with a delay function $\tau \in \mathcal{C}^\infty(S^1, \mathbb{R})$
- vary τ inside a finite dimensional subspace of $\mathcal{C}^\infty(S^1, \mathbb{R})$
- need more conditions on τ

Summary

Using sc-calculus and the M-polyfold IFT, we proved existence and local uniqueness of 1-periodic orbits of the delay equation

$$\dot{x}(t) = X_t\big(x(t-\tau)\big)$$

for small τ near an orbit of the vector field X. Moreover, the parametrization by delay is smooth.

This suggests that polyfold theory may be a powerful tool also for other questions from the field of differential delay equations.