Periodic delay orbits and the polyfold IFT

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Periodic delay orbits and the polyfold IFT

- existence result for a family of periodic delay orbits
- first example for the use of polyfold theory in the field of differential delay equations

Plan for this talk:

- Setting and main result
- Strategy
- Polyfold theory
- Some words about the proof
- Generalizations
Setting

$X : S^1 \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ smooth, time-dependent vector field on $\mathbb{R}^n$

(1-periodic in time)

$\tau \in \mathbb{R}$ parameter, called "delay"

Consider the following equation:

$$\dot{x}(t) = X_t(x(t - \tau))$$

We look for 1-periodic solutions $x : S^1 \rightarrow \mathbb{R}^n$. 
Consider the following equation:

\[ \dot{x}(t) = X_t(x(t - \tau)) \]

Remarks:

- This equation only makes sense in \( \mathbb{R}^n \). On a manifold, one may consider more complicated delay equations.
- The equation is non-local: \( \dot{x}(t) \) depends on the “past” of the trajectory \( x \). Thus, everything is more complicated than for ordinary differential equations. Solutions do not form a flow on \( \mathbb{R}^n \), rather a semi-flow on a function space.
- The equation is not smooth in \( \tau \).

For \( \tau = 0 \) we recover orbits of the vector field \( X \).

We vary the delay \( \tau \) and study 1-periodic solutions.
Theorem (Albers–S.)

Assume that $x_0$ is a non-degenerate 1-periodic orbit of the vector field $X$.

Then for every small enough $\tau \in \mathbb{R}$ there exists a (locally unique) solution $x_\tau \in C^\infty(S^1, \mathbb{R}^n)$ of the delay equation

$$\dot{x}(t) = X_t(x(t - \tau)).$$

The parametrization $\tau \mapsto x_\tau$ is smooth.

We generalize this to

- finitely many discrete delays
- suitable delay equations on manifolds
- certain cases of time-dependent delay (work in progress).
Solutions are zeros of the map

\[ s : \mathbb{R} \times C^\infty(S^1, \mathbb{R}^n) \longrightarrow C^\infty(S^1, \mathbb{R}^n) \]

\[ (\tau, x) \longmapsto \dot{x} - X(x(\cdot - \tau)). \]

We start with \( s(0, x_0) = 0 \) and claim that the zero set near \((0, x_0)\) carries the structure of a 1-dimensional smooth manifold (and is not contained in \(\{0\} \times C^\infty(S^1, \mathbb{R}^n)\)).

\[ \implies \] want to use an implicit function theorem (IFT)

Problem: The shift map \((\tau, x) \longmapsto x(\cdot - \tau)\) is not smooth.

\[ \implies \] use \(sc\)-calculus and the \(M\)-polyfold IFT instead
Classical differentiability

From now on denote $H_m := W^{m,2}(S^1, \mathbb{R}^n)$, in particular $H_0 = L^2(S^1, \mathbb{R}^n)$. This gives a sequence of compact and dense embeddings

$$
\cdots \hookrightarrow H_{m+1} \hookrightarrow H_m \hookrightarrow \cdots \hookrightarrow H_1 \hookrightarrow H_0.
$$

For $\tau \in \mathbb{R}$ and $x \in H_0$ we define the shift of $x$ by $\tau$ as follows:

$$
\varphi(\tau, x) := x(\cdot - \tau) \in H_0.
$$

Note: If $x \in H_m$, then $\varphi(\tau, x) \in H_m$ with $\|\varphi(\tau, x)\|_{H_m} = \|x\|_{H_m}$. 

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Fact

- The shift map

\[ \varphi : \mathbb{R} \longrightarrow \mathcal{L}(H_0, H_0) \]
\[ \tau \longmapsto (x \mapsto x(\cdot - \tau)) \]

is not continuous when the target space carries the operator topology.

- The shift map

\[ \varphi : \mathbb{R} \times H_1 \longrightarrow H_0 \]

is continuously differentiable with

\[ d\varphi(\tau, x)(T, \dot{x}) = \varphi(\tau, \dot{x}) - T \cdot \varphi(\tau, \dot{x}). \]
The solution space is cut out by the map

\[ s : \mathbb{R} \times H_1 \rightarrow H_0 \]

\[ (\tau, x) \mapsto \dot{x} - X(\varphi(\tau, x)) \]

From the properties of the shift map \( \varphi \):

- \( s \) is continuously differentiable.
- It is not smooth on any fixed level \( \mathbb{R} \times H_m \).

This means:

- We can use the \( C^1 \)-version of the IFT for Banach spaces and get a \( C^1 \)-manifold of smooth solutions \( (x_\tau)_\tau \).
- This can only provide a \( C^1 \)-parametrization \( \tau \mapsto x_\tau \), not smooth.
- If as domain we take \( \mathbb{R} \times H_2 \), we gain another derivative of \( s \), but we lose the Fredholm property.

But: The setting of sc-calculus comes naturally and solves the problem!
Polyfold theory

- was developed by H. Hofer, K. Wysocki and E. Zehnder
- is used for the study of moduli spaces of J-holomorphic curves
- was made to overcome the following problems:
  - varying domains
  - varying automorphism groups
  - transversality issues
  - non-smoothness of reparametrization maps

All details can be found in the book “Polyfold and Fredholm theory” by HWZ which is available on Arxiv.
Main ingredients of polyfold theory:

- **sc-Banach spaces** \( E = (E_0 \supseteq E_1 \supseteq E_2 \supseteq \ldots) \)
- **sc-differentiability**: new sense of differentiability for maps between sc-Banach spaces
- **M-polyfolds**: topological spaces that locally look like open subsets of sc-retracts in these sc-Banach spaces, with sc-smooth transition maps
- **polyfolds**: additionally allowing for orbifold behaviour

Important results:

- chain rule for sc-differentiability
- an implicit function theorem (IFT) for sc-smooth sc-Fredholm sections
Theorem (IFT for M-polyfold bundles)

Assume that

- we have a tame strong M-polyfold bundle admitting sc-smooth bump functions,
- $f$ is a sc-smooth section,
- $f$ has the sc-Fredholm property and
- $f$ is in good position.

Then the zero set of $f$ is a smooth manifold of dimension equal to the Fredholm index of $f$. 
Applying the M-polyfold IFT

sc-spaces:

\[
H = (H_m = W^{m,2}(S^1, \mathbb{R}^n))_{m \geq 0}
\]

\[
H^1 = (H^1_m = W^{m+1,2}(S^1, \mathbb{R}^n))_{m \geq 0}
\]

\(\mathbb{R}\) with constant sc-structure

trivial strong bundle:

\[
\mathbb{R} \times H^1 \rhd H \longrightarrow \mathbb{R} \times H^1
\]

section:

\[
s : \mathbb{R} \times H^1 \longrightarrow H
\]

\[
(\tau, x) \longmapsto \dot{x} - X(\varphi(x, \tau))
\]
Applying the M-polyfold IFT

Theorem (IFT for M-polyfold bundles)

Assume that

- we have a tame strong M-polyfold bundle admitting sc-smooth bump functions,
  our setting: the trivial sc-Hilbert space bundle
  \[ \mathbb{R} \times H^1 \ni (\tau, x) \mapsto \mathbb{R} \times H^1 \]
- \( f \) is a sc-smooth section,
  our setting: \( s : \mathbb{R} \times H^1 \to H, \ s(\tau, x) = \dot{x} - X(\varphi(x, \tau)) \)
- \( f \) has the sc-Fredholm property and some work!
- \( f \) is in good position.
  need transversality

Then the zero set of \( f \) is a smooth manifold of dimension equal to the Fredholm index of \( f \).
Indeed, we could prove the following:

**Theorem (Albers–S.)**

1. $s : \mathbb{R} \times H^1 \rightarrow H$ is a sc-Fredholm section of index 1.
2. If $x_0$ is a non-degenerate periodic orbit of $X$, then
   $ds(0, x_0) : \mathbb{R} \times H_1 \rightarrow H_0$ is surjective.

About the proof:

1. sc-smoothness of $s$ is immediate from sc-smoothness of the shift map $\varphi$ (which was shown by Frauenfelder–Weber). The sc-Fredholm property requires some work; we use the criteria given by Katrin Wehrheim.

2. More precisely, we show that $ds(0, x_0)(0, \cdot) : H_1 \rightarrow H_0$ is surjective if and only if $x_0$ is non-degenerate. This also implies that the 1-dimensional local solution set inside $\mathbb{R} \times H^1$ projects to an interval around $0 \in \mathbb{R}$. 
This together with the M-polyfold IFT implies our main theorem:

**Theorem (Albers–S.)**

Assume that $x_0$ is a non-degenerate 1-periodic orbit of the vector field $X$. Then for every small enough $\tau \in \mathbb{R}$ there exists a (locally unique) solution $x_\tau \in C^\infty(S^1, \mathbb{R}^n)$ of the delay equation

$$\dot{x}(t) = X_t(x(t - \tau)).$$

The parametrization $\tau \mapsto x_\tau$ is smooth.
1) finitely many discrete delays

- equations of the form $\dot{x}(t) = X_t(x(t - \tau_1), \ldots, x(t - \tau_k))$
  with $k$ delays $\tau_1, \ldots, \tau_k \in \mathbb{R}$ and $X : S^1 \times (\mathbb{R}^n)^k \to \mathbb{R}^n$
- by diagonal embedding, such an $X$ defines a vector field on $\mathbb{R}^n$, and we can start with a non-degenerate orbit $x_0$ of this vector field
- get a $k$-dimensional solution manifold
2) suitable delay equations on manifolds

- e.g. equations of the form $\dot{x}(t) = f_t(x(t - \tau))X_t(x(t))$
- instead of linear sc-Hilbert spaces we get a sc-Hilbert manifold and a non-trivial bundle
- to insure that non-degeneracy of $x_0$ implies surjectivity we need to assume that $x_0^*TM$ is trivial
- get a 1-dimensional solution manifold as before
3) certain cases of time-dependent delay in $\mathbb{R}^n$ (work in progress)

- equations of the form $\dot{x}(t) = X_t(x(t - \tau(t)))$
  with a delay function $\tau \in C^\infty(S^1, \mathbb{R})$
- vary $\tau$ inside a finite dimensional subspace of $C^\infty(S^1, \mathbb{R})$
- need more conditions on $\tau$
Using sc-calculus and the M-polyfold IFT, we proved existence and local uniqueness of 1-periodic orbits of the delay equation

\[ \dot{x}(t) = X_t(x(t - \tau)) \]

for small \( \tau \) near an orbit of the vector field \( X \). Moreover, the parametrization by delay is smooth.

This suggests that polyfold theory may be a powerful tool also for other questions from the field of differential delay equations.