Disk counting via family Floer theory

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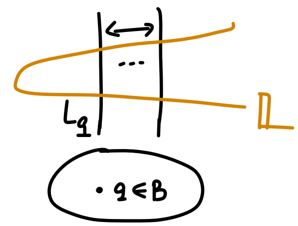
- **Background** Family Floer, Fukaya 2001, Floer homology for families of Lagrangians.

to SYZ picture:

$$X \downarrow B$$



$\boxed{HF(\mathbb{L}, L_q),\ q \in B \text{ simultaneously}}$ ⟿ "coherent sheaf"

"family Floer functor"

- Abouzaid, "Homological Mirror Symmetry without corrections."

Goal: try to include quantum/instanton corrections. Related to:

Conjecture (Auroux)
(X, ω, J) Kähler manifold, D: anticanonical divisor, Ω: holo volume form on $X \setminus D$.
Then, a mirror space $X^\vee_{\mathbb{C}}$ = moduli of special Lag. tori in $X \setminus D$ equipped with flat $U(1)$-connections.
"$L \longleftrightarrow H^1(L; U(1))$"
$W^\vee_{\mathbb{C}}: X^\vee_{\mathbb{C}} \longrightarrow \mathbb{C}$ given by FOOO's m_0-obstruction to Floer homology

My thesis gives a positive answer but in a non-archimedean setting.

Theorem (non-archimedean SYZ mirror construction) Let $X_0 \subseteq X$.

Suppose we have a reasonable (e.g. special Lag) smooth Lagrangian torus fibration $\pi: X_0 \longrightarrow B_0$

We can naturally construct

- A rigid analytic space $X^\vee$ over the Novikov field Λ
- A global superpotential $W^\vee: X^\vee \longrightarrow \Lambda$
- A dual fibration $\pi^\vee: X^\vee \longrightarrow B_0$

unique up to isomorphism, such that for $q \in B_0$, the dual fiber is $(\pi^\vee)^{-1}(q) \overset{\text{set}}{=\!=} H^1(L_q; U_\Lambda) \cong U_\Lambda^n$

- The dual fiber is now $H^1(L; U_\Lambda)$ in place of $H^1(L; U(1))$
  $U_\Lambda = \{$ norm-one elements in the Novikov field $\Lambda \}$ resembles $U(1)$ "non-archimedean torus"
- $X^\vee \overset{\text{set}}{=\!=} \coprod_{q \in B_0} H^1(L_q; U_\Lambda) \cong B_0 \times U_\Lambda^n$ (extra rigid analytic space structure $\Leftarrow$ wall crossing)

Examples $\begin{cases} \text{Toric fibration} \\ \text{Gross fibration} \star \end{cases}$ Today application: disk counting

① The rigid analytic space structure on $X^\vee$ is locally modeled on <u>the non-archimedean torus fibration</u>:

$$\mathrm{trop} = \mathrm{val}^n : (\Lambda^*)^n \longrightarrow \mathbb{R}^n, \quad (y_i) \longmapsto (\mathrm{val}(y_i))$$

analog of $\mathrm{Log} : (\mathbb{C}^*)^n \longrightarrow \mathbb{R}^n$

Namely, if $\Delta \overset{\text{small}}{\subseteq} B_0$, then $(\pi^\vee)^{-1}(\Delta) \simeq \mathrm{trop}^{-1}(\Delta)$ $\boxed{\pi^\vee \overset{\text{loc}}{=\!=} \mathrm{trop}}$

<u>Think</u> $\Delta \hookrightarrow \mathbb{R}^n$ up to $GL(n,\mathbb{Z})$-transformations.

integral affine structure on B_0

$$\mathrm{trop}^{-1}(0) = (\mathrm{val}^{-1}(0))^n = U_\Lambda^n$$

$$\mathrm{trop}^{-1}(\vec{c}) = \{\mathrm{val}(y_i) = c_i\} = T^{c_1}U_\Lambda \times \cdots \times T^{c_n}U_\Lambda$$

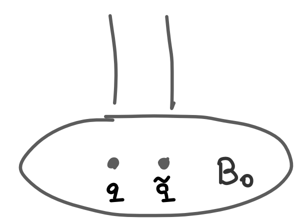
$$\Longrightarrow \quad \mathrm{trop}^{-1}(0) \simeq \mathrm{trop}^{-1}(\vec{c})$$

$$y_i \longleftrightarrow T^{c_i} y_i$$

<u>Recall</u> $X^\vee \overset{\text{set}}{=\!=} \bigsqcup H^1(L_q; U_\Lambda) \simeq B_0 \times U_\Lambda^n$

gluing map

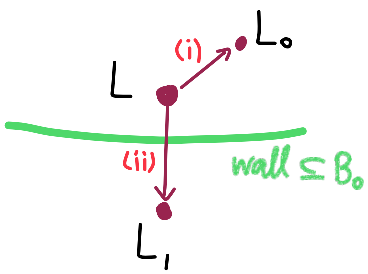
$$H^1(L; U_\Lambda) \equiv U_\Lambda^n \overset{\phi}{\longleftrightarrow} U_\Lambda^n \equiv H^1(\tilde{L}; U_\Lambda)$$



<u>We can prove</u> (rigid analytic geometry)

$$\phi : y_i \longleftrightarrow T^{c_i} y_i \exp(F_i(y))$$

A gluing map Φ for $(X^\vee, W^\vee)$



(i) $H^1(L; U_\Lambda) \cong H^1(L_0; U_\Lambda)$ $\quad y_i \longleftrightarrow T^{c_i} y_i$ just like $\mathrm{trop}^{-1}(0) \cong \mathrm{trop}^{-1}(c)$

(ii) $H^1(L; U_\Lambda) \cong H^1(L_1; U_\Lambda)$ $\quad y_i \longleftrightarrow T^{c_i} y_i \exp(F_i(y))$ → quantum corrections (Maslov-zero disks)

Note: Knowing this pattern is important for application
The proof requires non-archimedean analysis

② The local expressions of $W^\vee$

$$W^\vee|_L \overset{\text{def}}{=} \sum_{\mu(\beta)=2} T^{E(\beta)} y^{\partial\beta} \underset{= \text{ the open GW}}{n_\beta} \in \Lambda[[\pi_1(L)]] \cong \Lambda[[y_1^{\pm} \dots y_n^{\pm}]]$$

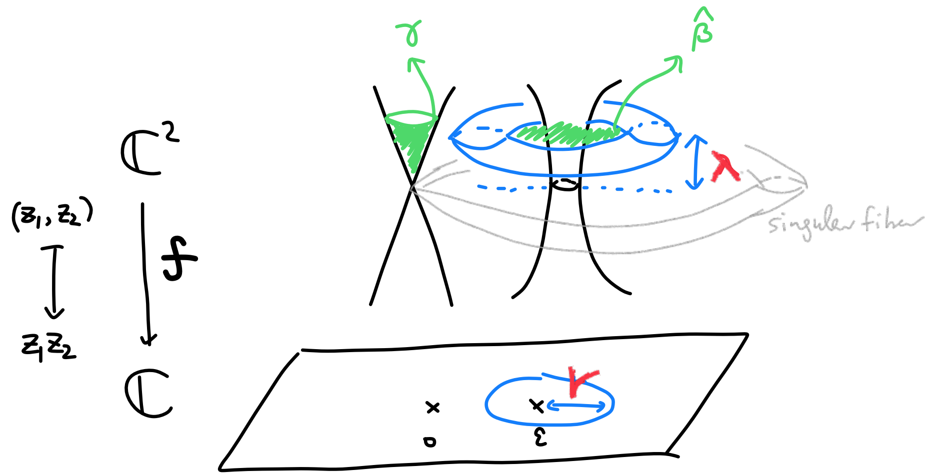
By Main Theorem, the gluing maps must match various local expressions of $W^\vee$

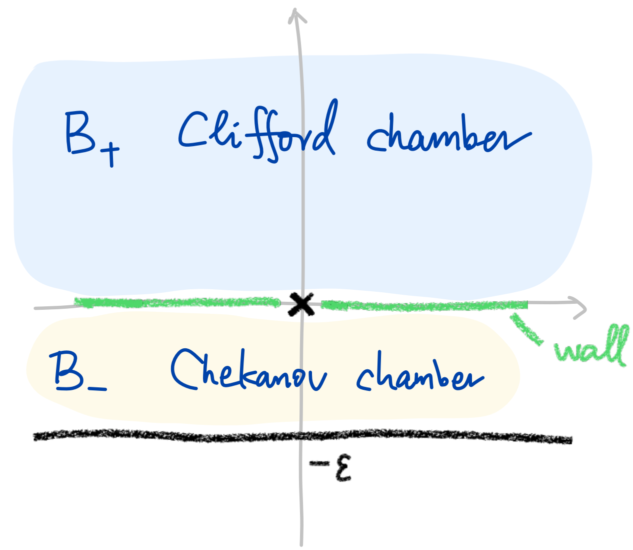
(i) $W^\vee|_L \longleftrightarrow W^\vee|_{L_0}$ $\quad y_i \longleftrightarrow T^{c_i} y_i$ $\Rightarrow n_\beta^L = n_\beta^{L_0}$

(ii) $W^\vee|_L \longleftrightarrow W^\vee|_{L_1}$ $\quad y_i \longleftrightarrow T^{c_i} y_i \exp(F_i(y))$ $\Longrightarrow$? application

Application (Disk counting)

$\boxed{n=2}$ Gross's fibration on $X = \mathbb{C}^2$, $(z_1, z_2) \overset{\pi}{\longmapsto} \Big(\underbrace{\tfrac{1}{2}|z_1|^2 - \tfrac{1}{2}|z_2|^2}_{\lambda},\ \underbrace{|z_1 z_2 - \varepsilon| - \varepsilon}_{r-\varepsilon}\Big) \in B_0$



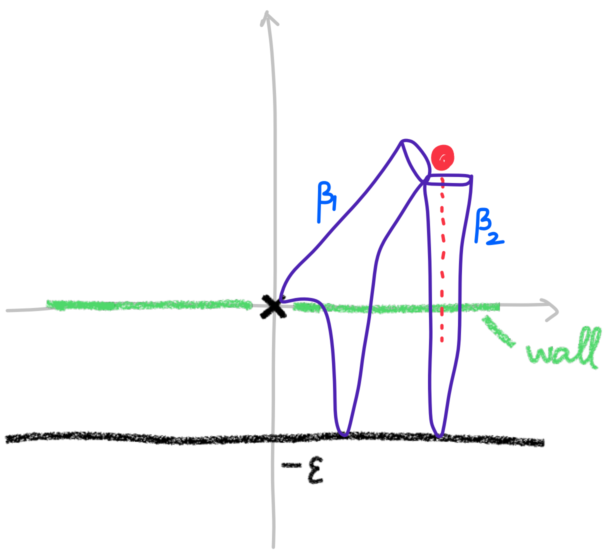


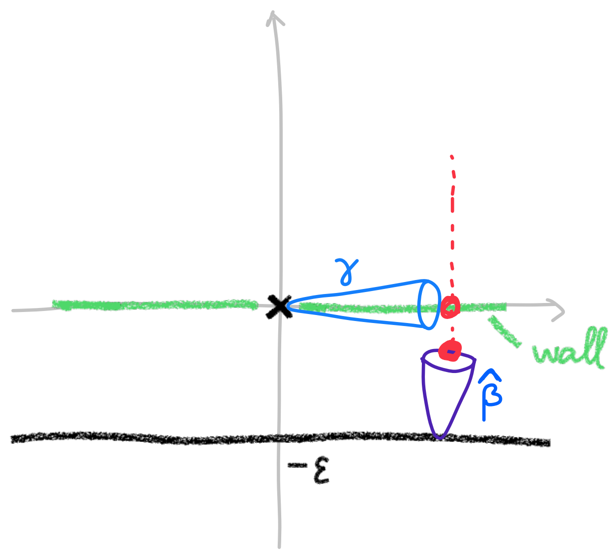
$r > \varepsilon$: Clifford

$r < \varepsilon$: Chekanov

$\mu(\hat{\beta}) = 2$; $\mu(\gamma) = 0$

$B_0 = B_+ \cup B_- \cup$ "wall"





$\pi_2(X, L_0) \cong \pi_2(X, L_1)$

slightly abusing the notations.

$$\beta_1 = \hat\beta + \gamma$$

$$\beta_2 = \hat\beta$$

Clifford
$$W_+ = T^{E(\beta_1)} Y^{\partial\beta_1} + T^{E(\beta_2)} Y^{\partial\beta_2}$$
$$= T^{E(\hat\beta)} Y^{\partial\hat\beta} \left(1 + T^{E(\gamma)} Y^{\partial\gamma}\right)$$

Chekanov
$$W_- = T^{E(\hat\beta)} Y^{\partial\hat\beta}$$

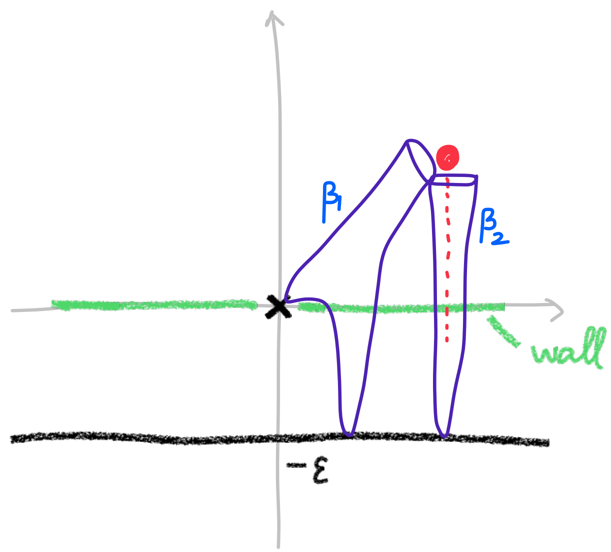
- $n_{\beta_1} = n_{\beta_2} = 1$
- $n_{\hat\beta} = 1$

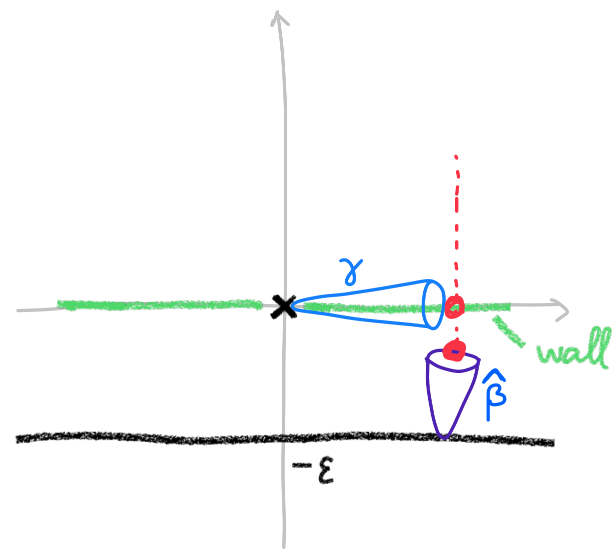
} well-known open GW

$$\boxed{\Phi(W_+) = W_-}$$

gluing map $Y_i \longleftrightarrow T^{c_i} Y_i \exp(F_i(Y))$.

(may assume $c_i = 0$)





$$\beta_1 = \hat\beta + \gamma$$

$$\beta_2 = \hat\beta$$

where $\mathcal{H} = [\mathbb{CP}^1]$

When we compactify $\mathbb{C}^2$ to $\mathbb{CP}^2$, a Clifford-type torus L bounds new $\beta_3 = \mathcal{H} - \beta_1 - \beta_2$

($n_{\beta_3} = 1$ by Cho–Oh)

$$\overline{W}_+ = W_+ + T^{E(\mathcal{H}-\beta_1-\beta_2)} Y^{-\partial\beta_1 - \partial\beta_2}$$

$$\overline{W}_- = W_- + \textbf{?}$$

Observation : No new Maslov-zero disk

$$\Phi(\overline{W}_+) = \overline{W}_-$$

the same gluing map

Set $Y_1 = Y^{\partial\gamma}$; $Y_2 = Y^{\partial\hat{\beta}}$. Then $\Lambda[[\pi_1(L)]] \cong \Lambda[[Y_1^{\pm}, Y_2^{\pm}]]$

$$\phi(W_+) = W_- \Longrightarrow \begin{cases} \phi(Y_1) = T^{c_1} Y_1 \\ \phi(Y_2) = T^{c_2} Y_2 \left(1 + T^{E(\gamma)} Y_1\right)^{-1} \end{cases}$$

$$\phi(\overline{W}_+) = \overline{W}_- \Longrightarrow$$

W_+ and $\overline{W}_+$ are known by Cho-Oh

compute $\overline{W}_-$ (i.e. compute open GW)

Outcome

$$\overline{W}_- = T^{E(\hat{\beta})} Y^{\partial\hat{\beta}} + T^{E(\mathcal{H}-2\hat{\beta}-\gamma)} Y^{-2\partial\hat{\beta}-\partial\gamma} + T^{E(\mathcal{H}-2\hat{\beta}+\gamma)} Y^{-2\partial\hat{\beta}+\partial\gamma} + \mathbf{2}\, T^{E(\mathcal{H}-2\hat{\beta})} Y^{-2\partial\hat{\beta}}$$

$\Longrightarrow$ open GW : 1, 1, 1, 2 agree with Chekanov-Schlenk e.g. $n_{\mathcal{H}-2\hat{\beta}} = 2$

without explicitly finding the holo. disks.

This idea can be applied in higher dimensions as well :

(compactification of $\mathbb{C}^n$)

Theorem For a Chekanov-type Lagrangian torus L in $\mathbb{CP}^n$, we can compute all the non-trivial open Gromov-Witten invariants. Indeed, we can compute its potential function :

(rely on the component of wall)

(Notation : Maslov-zero classes : $\gamma_1, \cdots, \gamma_{n-1}$. We set $\gamma = \gamma_1 + \cdots + \gamma_{n-1}$. We also define $H = [\mathbb{CP}^1]$)

$$\bar{W}_- = T^{E(\hat{\beta})} Y^{\partial\hat{\beta}} + T^{E(H - n\hat{\beta} - \gamma)} Y^{-n\partial\hat{\beta} - \partial\gamma} \left(1 + T^{E(\gamma_1)} Y^{\partial\gamma_1} + \cdots + T^{E(\gamma_{n-1})} Y^{\partial\gamma_{n-1}}\right)^n \quad \text{over } \Lambda$$

Pascaleff - Tonkonog

Theorem 1.4 *(Corollary 5.8). For each $1 \le k \le n$, $\mathbb{C}P^n$ contains a monotone Lagrangian torus whose potential is given by*

$$W = \sum_{i=k}^{n} x_i^{-1} + (x_k)^k \cdot \left(\sum_{i=1}^{k} x_i^{-1}\right)^k \cdot \prod_{i=1}^{n} x_i. \qquad (1.6)$$

Their work is more general, but when $k = n$, we observe

$$W = \frac{1}{x_n} + x_n^n \left(\frac{1}{x_1} + \cdots + \frac{1}{x_n}\right)^n x_1 \cdots x_n$$

$$= \frac{1}{x_n} + \left(1 + \frac{x_n}{x_1} + \cdots + \frac{x_n}{x_{n-1}}\right)^n \frac{x_1}{x_n} \cdots \frac{x_{n-1}}{x_n} \cdot x_n^n$$

Note $\frac{1}{x_n} \longleftrightarrow Y^{\partial\hat{\beta}}$; $\frac{x_n}{x_k} \longleftrightarrow Y^{\partial\gamma_k}$

Further examples :

X	$\bar{X}$
$\mathbb{C}^2$	$\mathbb{CP}^2$
$\mathbb{C}^n$	$\mathbb{CP}^n$, $\mathbb{CP}^r \times \mathbb{CP}^{n-r}$ (Done) **or** a projective toric compactification of $\mathbb{C}^n$ (In progress)

More generally

perhaps any toric Calabi-Yau

(c.f. Chan-Lau-Leung,
Abouzaid-Auroux-Katzarkov, etc.)

Also have Gross fibration, Clifford, Chekanov tori

Our method seems different from Pascaleff-Tonkonog's work.

I believe there should be something interesting to explore