The homotopy of the contactomorphisms of a 3-fold. The overtwisted mirage.

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The notation

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The usual suspect.

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- Our main result says that up to the action of $\mathrm{Cont}(M^3,\xi))$ in the space of Darboux balls, the computation of $\mathrm{Cont}(M^3,\xi)$ is pure algebraic topology.
- The key actor is: Eliashberg result on the contractibility of the contactomorphisms in the ball.

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- $\mathbb{S}^3$ Eliashberg.
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- Lens spaces, thickened torus, solid torus, Giroux-Honda.
- Finiteness of the representable homotopy classes Colin, Giroux, Honda.

$\pi_O(Cont(M,\xi))$ and $\pi_1(\mathfrak{C}Str(M,\xi))$ (II)

Definition

A formal contact structure, in dimension 3, is just a codimension 1 distibution.
A formal contactomorphism (ϕ, F_s) is a pair given by a diffeomorphism $\phi : M \to M$ and a formal derivative $F_s : TM \to \phi^* TM$ such that F_s, $s \in [0,1]$, is an isomorphism of bundles, $F_0 = d\phi$, $F_1(\xi) = \xi$.

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- First result. Bourgeois (2006) (also Geiges- Gonzalo): The space of contact structures of T^3 contains a homotopically non-trivial loop that is no torsion in $\pi_1(\mathfrak{C}Str(T^3, \xi)$, i.e. $\mathbb{Z} \subset \pi_1(\mathfrak{C}Str(T^3, \xi))$. Equivalently, it contains a non-isotopic to the identity contactomorphism that is formally contact isotopic to the identity.

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- Bourgeois example provides an SFT invariant for any loop of contact structures, more in general for any sphere of contact structures. Giroux et al. push convex surfaces to the limit.

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- Bourgeois example provides an SFT invariant for any loop of contact structures, more in general for any sphere of contact structures. Giroux et al. push convex surfaces to the limit.
- Remark: Bourgeois example is formally trivial and geometrically non-trivial (a contradiction with my first slide). Our main result says that up to the action of $\text{Cont}(M^3)$ in the space of Darboux balls, the computation of $\text{Cont}(M^3,\xi)$ is pure algebraic topology.

The main theorem

Denote $D_k := Im(\pi_k(\mathrm{FCont}(M,\xi,\mathrm{rel}\, p)) \to \pi_k(Diff(M,\mathrm{rel}\, p)))$.

Theorem

Let (M,ξ) be any compact tight contact 3–manifold. Consider the inclusion

$$i_D : \mathrm{Cont}(M,\xi;\mathrm{rel}\, p) \hookrightarrow \mathrm{Diff}(M;\mathrm{rel}\, p).$$

The following holds:

1. *The homomorphisms $\pi_k(i_D)$ are injective for any $k \geq 1$.*
2. *The inclusion $i_F : \mathrm{Cont}(M,\xi;\mathrm{rel}\, p) \hookrightarrow \mathrm{FCont}(M,\xi;\mathrm{rel}\, p)$ is injective at the level of path–components.*
3. *The image of $\pi_k(i_D)$ is precisely D_k for any k.*

Corollaries.

- The connected component of the Legendrian embeddings space for a given long knot type is homotopically equivalent to $K(G,1) \times U(2)$, where $G = \pi_1(\mathfrak{Emb}_{N,jN}(\mathbb{S}^1, \mathbb{S}^3))$ is the fundamental group of the corresponding component of the space of long embeddings.

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- Several computations of homotopy types of contactomorphisms groups of tight manifold $\text{Cont}(\mathbb{S}^1 \times \mathbb{S}^2, \xi_{\text{std}}) \simeq \text{U}(1) \times \Omega(U(1)) \times \mathbb{S}^1$.

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- More to come.

The oracle (I)

Hatcher is the architect of the computation of the smooth case.

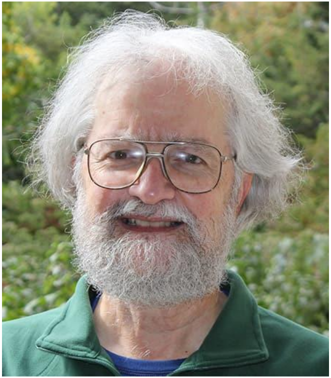

Figure: The oracle.

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Theorem (3-skeleton)

The space of diffeomorphisms of $\mathbb{B}^3$ is contractible.

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$$\begin{array}{ccc} \mathrm{Diff}(H^{g-1}) & \hookrightarrow & \mathrm{Diff}(H^g)) \\ & & \downarrow \\ & & \mathfrak{Emb}^{\partial \mathbb{D}^2}(\mathbb{D}^2, H^g) \end{array}$$

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$$\begin{array}{ccc} \mathrm{Diff}(C_\gamma) & \hookrightarrow & \mathrm{Diff}(\mathbb{B}^3)) \\ & & \downarrow \\ & & \mathfrak{Emb}^0_{N,jN}(\mathbb{S}^1, \mathbb{S}^3) \end{array}$$

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Theorem (1-skeleton)

The path–connected component $\mathfrak{Emb}^0_{N,jN}(\mathbb{S}^1, \mathbb{S}^3)$ of the space of smooth long embeddings containing the long unknot is contractible.

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Theorem (1-skeleton)

The path–connected component $\mathfrak{Emb}^0_{N,jN}(\mathbb{S}^1, \mathbb{S}^3)$ of the space of smooth long embeddings containing the long unknot is contractible.

Proof: We have to compute the homotopy of $\text{Diff}(C_\gamma)$ Take a handle decomposition $C_\gamma \simeq H^2 \cup H^3$. Fix a sphere of diffeomorphisms $\phi_z : C_\gamma \to C_\gamma$, $z \in \mathbb{S}^k$. Compute $\phi_z(H^2) = H^2_z$, by Theorem "2-skeleton", it is a contractible family. Thus, by isotopy extension Theorem, there exists a family of flows $\Psi_{z,t} :\in Diff(C_\gamma)$ such that they satisfy $\Psi_{z,1} \circ \phi_z(H^2) = H^2$. So we may assume that our family of diffeomorphisms has compact support on the ball H^3. We are done.

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Figure: Eliashberg and Mishachev.

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The group of compactly supported contactomorphisms of the 3-ball is contractible.

- The result was announced by Eliashberg 30 years ago. The written proof was working till $\pi_1(\mathrm{Cont}(\mathbb{B}^3))$, maybe $\pi_2(\mathrm{Cont}(\mathbb{B}^3))$. It is clear that you need Igusa theorem if you try to adapt the initial proof (and more things). Work in progress by Eliashberg and Mishachev.

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- We are tying to check an alternative argument in the last few months. I will explain it at the very end.

The consequences.

- We know how to deal with the 3-skeleton. But, what about the 2-skeleton?
- Very well-understood problem in Contact Topology: convex surface theory.
- What we need is a multi-parametric convex surface theory. It is good enough if it works for disks (being able to assume that the 2-cells are convex disks (all of them equal).

Theorem

The space of disks with boundary fixed on a Legendrian unknot on a tight 3-fold is homotopically equivalent to the space of smooth disks containing it.

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Let us try to prove it on a small Darboux ball with fixed boundary the standard convex sphere. I am assuming that I take the unknot in the boundary of the ball, then I just have the natural action.

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This implies that $\mathsf{Cont}(\mathbb{B}^3) \simeq *$ if and only if $\mathfrak{Emb}^{\mathsf{std}}(\mathbb{D}^2, \mathbb{B}^3) \simeq *$.

The consequences. (III)

Emmanuel Giroux makes his appearance. We want to show that this also works for any small ball with random boundary.

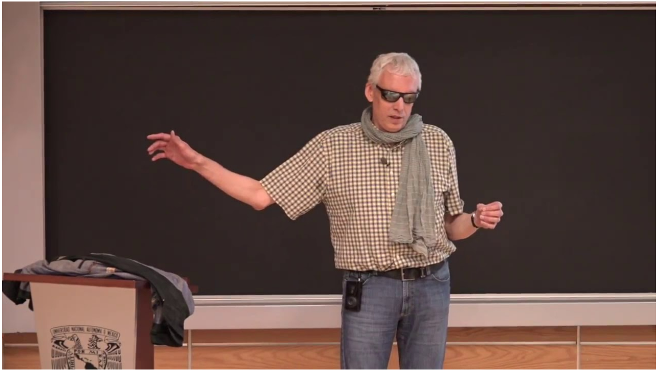

Figure: E. Giroux.

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By Giroux genericity theorem, the space of convex embeddings is C^∞-dense, by Giroux OT criterium its dividing set is a connected segment, by Giroux realization theorem, then there is a standard embedding C^0–close.

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By Giroux genericity theorem, the space of convex embeddings is C^∞-dense, by Giroux OT criterium its dividing set is a connected segment, by Giroux realization theorem, then there is a standard embedding C^0–close.
Ball case: Fix a family $e_{z,r}$ of embeddings of disks, $(z,r) \in \mathbb{B}^{k+1}$, such that they are standard for $(z,1) \in \mathbb{S}^k$. Then find, by the previous argument a northern hemisphere (a standard disk above all the family and in the interior of the ball. The same with the southern hemisphere. This provides a smaller standard sphere in which we apply the previous lemma.
Let us study a general family $e_{z,r}$ of standard disks on a general 3-fold. Fix a smooth ball $B_{z,r}$ around each disk of the family. Create a fibration with base $\mathbb{B}^{k+1}$ and fiber $\mathfrak{Emb}^{\text{std}}_{\gamma_0}(\mathbb{D}^2, \mathbb{B}^{z,r})$. This is a Serre fibration with contractible fiber. Thus the partial section $e_{z,1}$, extends to a global section.

The proof.(I)

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- Thus we start with a family of contactomorphisms ϕ_z, $z \in \mathbb{S}^2$ such that they lie on a family of diffeomorphisms $\phi_{z,r}$, $(z, r) \in \mathbb{B}^{k+1}$. We want to find a homotopic ball (relative to the boundary). To do it we try to push the image of the skeleton back to itself.

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- The 0-cells are ready, we have made the construction relative to them.

The proof.(I)

- The 1-cells can be easily deformed by the Theorem 1-**skeleton** to make all the 1-cells Legendrian. Once there, extension of isotopy for Legendrians pushes everything back to the original 1-skeleton.

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- The 2-cells can be push back to standard ones first. Then extension of isotopy for convex, with fixed foliation, surfaces pushes them back to the initial one.
- Thus, we are left with a family of contactomorphisms that are compactly supported on a set of 3-cells. Eliashberg-Mishachev theorem tells you that you can push back to the identity through contactactomorphisms.

Relative to a point?

Let us try to understand why the Bourgeois example does not work in our setup. Fix $(T^3(x,y,z), \ker(\alpha = \sin(2z)dx + \cos(2z)y))$. There are two knots that are $\gamma_0(t) = (t,0,0)$ and $\gamma_1(t) = (t,0,\pi)$.

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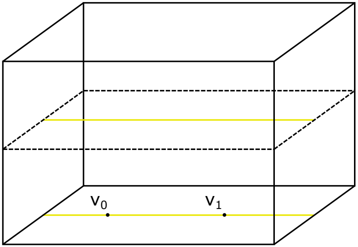

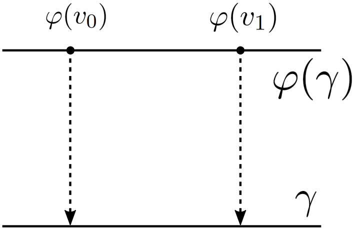

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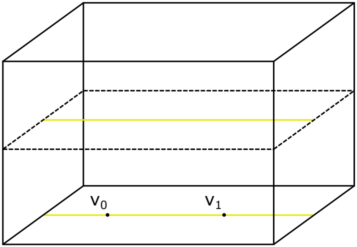

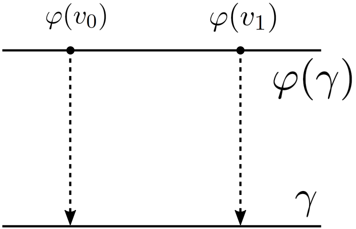

Create a Giroux triangulation that includes that knot. Just place two 0-cells v_0, v_1 and two 1-cells.

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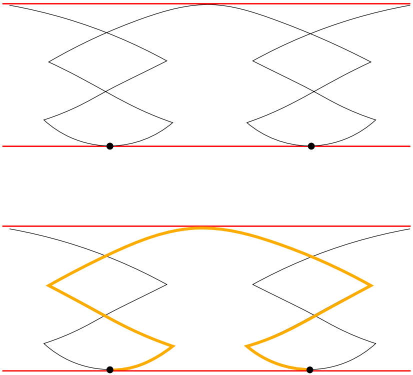

Up, in red two 1-cells and two 0-cells in red conforming the Gigghini Legendrian knot. See also, its image through a Hamiltonian isotopy pushing the 0–cells back to original position (clearly Legendrian isotopic). In gold the deformation from the point of view of a Legendrian arc, the formal class has changed.

The mirage.

- Let us try to prove that the argument works for a general contact (possibly OT) contact 3-fold.
- Key remark: you may check that small neighborhood of any cell is tight taking the triangulation small enough. We mean the initial triangulation, but also the images through any compact family of diffeos.
- To make our life simple, recall that $\mathrm{Cont}(H^g) = \mathit{Diff}(H^g) = \{*\}$. Moreover, any Giroux handle-body is a convex neighborhood of an open book page, thus it is tight. So nothing changes in the OT case. If we can pass the 1-skeleton, we are done.

The mirage.

- Something that we did not detail:actual proof of the 1-skeleton. Proof by picture.
- In geometric terms what we use is that hte natural inclusion of Legendrian unknots fixing 2 points in smooth unknots is a homotopy injection. This is the problem since we have that the Legendrian unknots are loose in an overtwisted contact manifold, therefore they do not inject in smooth unknots.

Tight $\implies$ OT

- Let us prove ELiashberg 89.

- 1- skeleton is conformed of loose arcs. So by Fuchs-Tabachnikov, we obtain that there is a homotopy equivalence between formal arcs and Legendrian arcs.

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- Use the obstruction theory method. Assume that you have a 3 cell that is OT (fix OT disk) and the rest is tight.

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Tight $\Longrightarrow$ OT

- Let us prove ELiashberg 89.
- Use the obstruction theory method. Assume that you have a 3 cell that is OT (fix OT disk) and the rest is tight.
- So, 0, 2 and 3 skeleton work with no changes.
- 1- skeleton is conformed of loose arcs. So by Fuchs-Tabachnikov, we obtain that there is a homotopy equivalence between formal arcs and Legendrian arcs.

Homotopy of the contactomorphism group of an OT manifold. (I)

Theorem (Work in progress.)

The space of overtwisted disks of a 3-fold is homotopically discrete. Equivalently the following two inclusion morphisms are homotopy equivalences on the connected component of the identity

$$\mathrm{Cont}(M^3, \xi, \mathrm{rel}\, \mathbb{D}_{OT}) \to \mathrm{Cont}(M^3, \xi, \mathrm{rel}\, p) \to \mathrm{FCont}(M^3, \xi, \mathrm{rel}\, p)$$

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Homotopy of the contactomorphism group of an OT manifold. (I)

Theorem (Work in progress.)

The space of overtwisted disks of a 3-fold is homotopically discrete. Equivalently the following two inclusion morphisms are homotopy equivalences on the connected component of the identity

$$\mathrm{Cont}(M^3,\xi,\mathrm{rel}\,\mathbb{D}_{OT}) \to \mathrm{Cont}(M^3,\xi,\mathrm{rel}\,p) \to \mathrm{FCont}(M^3,\xi,\mathrm{rel}\,p)$$

.

The equivalence is obvious from the fibration sequence

$$\begin{array}{ccc} \mathrm{Cont}(M^3,\xi,\mathrm{rel}\,\mathbb{D}_{OT}) & \longrightarrow & \mathrm{Cont}(M^3,\xi,\mathrm{rel}\,p) \\ & & \downarrow \\ & & Emb^{OT}(\mathbb{D}^2, M^3, \xi, \mathrm{rel}\,p) \end{array}$$

Homotopy of the contactomorphism group of an OT manifold. (II)

- Proof for a particular manifold: Denote by $(\mathbb{S}^3, \xi_0)$ the unique overtwisted contact structure formal contact equivalent to the tight one. Assume that it is built from a full Lutz twist around the standard transverse unknot in $(\mathbb{S}^3, \xi_{\text{std}})$. Take a sphere transverse to the full Lutz twist (intersecting along two disks). Check that removing such a sphere in $\mathbb{S}^3$ we get two balls that are overtwisted at the boundary. Moreover, the global manifold is defined as the manifold obtained by removing a small neighborhood of a double OT disk, whose boundary is clearly an overtwisted standard sphere.

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- We have to compute the following diagram

$$\begin{array}{ccc} \mathrm{Cont}(B^3, \xi_0 \,\mathrm{rel}\, \mathbb{D}_{OT}) \times \mathrm{Cont}(B^3, \xi_0 \,\mathrm{rel}\, \mathbb{D}_{OT}) & \longrightarrow & \mathrm{Cont}(\mathbb{S}^3, \xi_0, \mathrm{rel}\ p) \\ & & \downarrow \\ & & Emb^{OT}(\mathbb{S}^2, \mathbb{S}^3, \xi_0, \mathrm{rel}\ p) \end{array}$$

Thanks a lot for listening.