# On the spatial restricted three-body problem

Agustin Moreno

Uppsala Universitet

Joint with Otto van Koert: arXiv:2011.10386, arXiv:2011.06562.
Spin-off (holomorphic dynamics): arXiv:2011.06568.
Survey available: arXiv:2101.04438.

# Spatial circular restricted three-body problem

**Setup.** Three objects: Earth (E), Moon (M), Satellite (S) with masses $m_E, m_M, m_S$, under gravitational interaction.

Classical assumptions:

1. **(Restricted)** $m_S = 0$, i.e. $S$ is *negligible.*
2. **(Circular)** The *primaries* $E$ and $M$ move in circles around their center of mass.
3. **(Planar)** $S$ moves in the plane spanned by $E$ and $M$.

**Spatial case**: drop the planar assumption.

**Goal:** Study motion of $S$.

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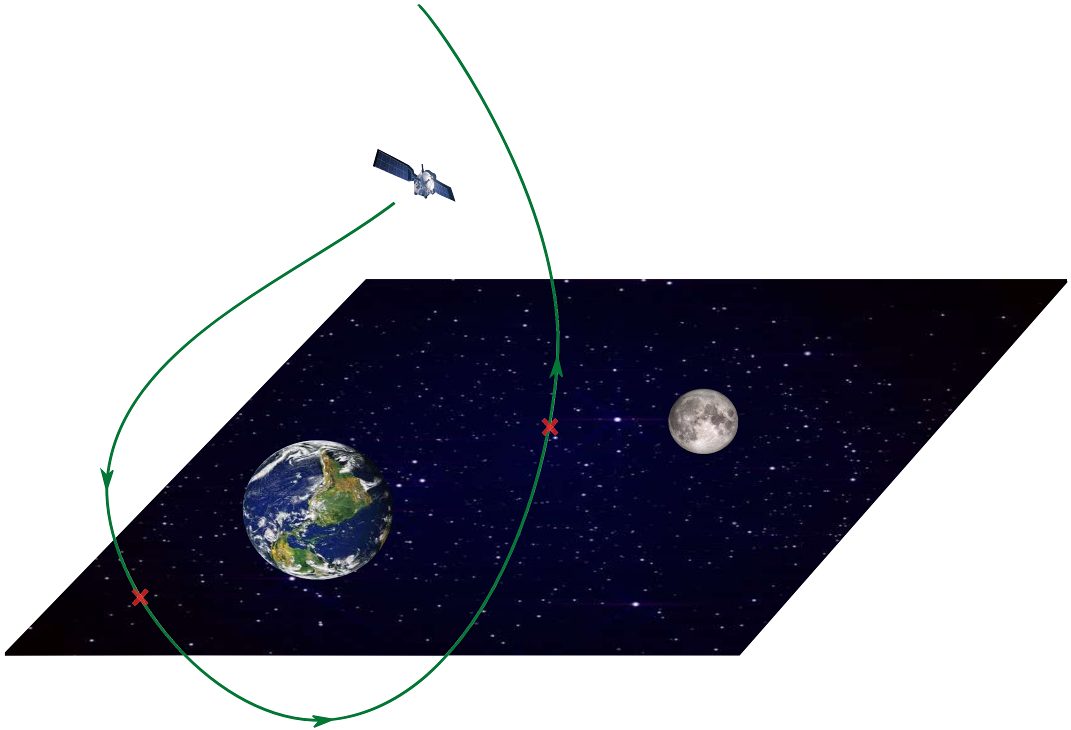

# Spatial circular restricted three-body problem

In rotating coordinates so that $E, M$ are fixed, the Hamiltonian is autonomous and so a conserved quantity:

$$H : \mathbb{R}^3 \backslash \{E, M\} \times \mathbb{R}^3 \to \mathbb{R}$$

$$H(q,p) = \frac{1}{2}\|p\|^2 - \frac{\mu}{\|q-M\|} - \frac{1-\mu}{\|q-E\|} + p_1 q_2 - p_2 q_1,$$

where we normalize so that $m_E + m_M = 1$, and $\mu = m_M$.

**Planar problem:** $p_3 = q_3 = 0$ (flow-invariant subset).

**Two parameters:** $\mu$, and $H = c$ Jacobi constant.

# Integrable limit cases

If $\mu = 0 \rightsquigarrow H = K + L$, where

$$K(q,p) = \frac{1}{2}\|p\|^2 - \frac{1}{\|q\|}$$

is the *Kepler energy* (two-body problem), and

$$L = p_1 q_2 - p_2 q_1$$

is the Coriolis/centrifugal term. This is the *rotating Kepler problem.*

**Fact:** $c \to -\infty \rightsquigarrow$ Kepler problem.

# Hill regions

$H$ has five critical points: $L_1, \ldots, L_5$ called *Lagrangians*.

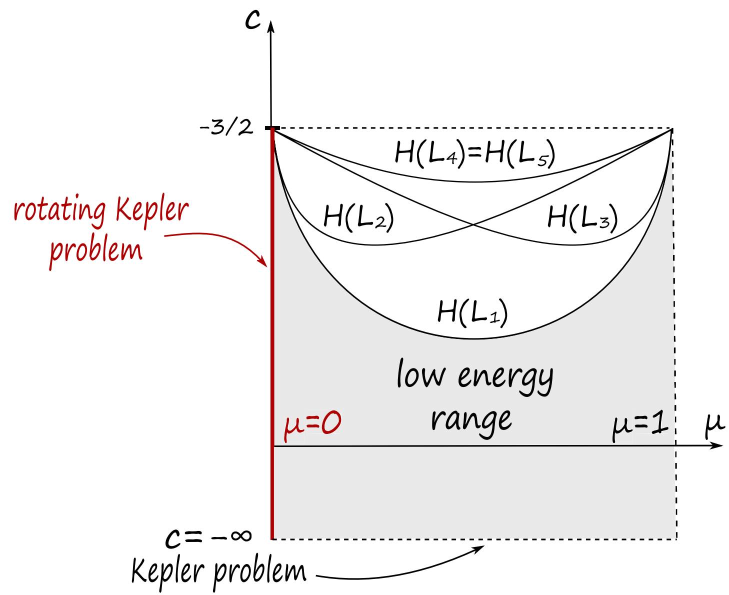


Figure: The critical values of $H$.

# Hill regions

For $c \in \mathbb{R}$, let $\Sigma_c = H^{-1}(c)$. Consider

$$\pi : \mathbb{R}^3 \backslash \{E, M\} \times \mathbb{R}^3 \to \mathbb{R}^3 \backslash \{E, M\}$$

$$(q, p) \mapsto q,$$

and the *Hill region*

$$\mathcal{K}_c = \pi(\Sigma_c).$$

# Low energy Hill regions

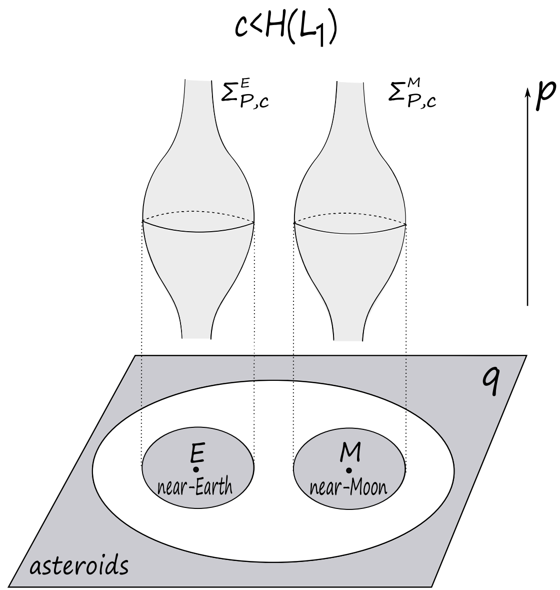


Figure: Morse theory in the three-body problem.

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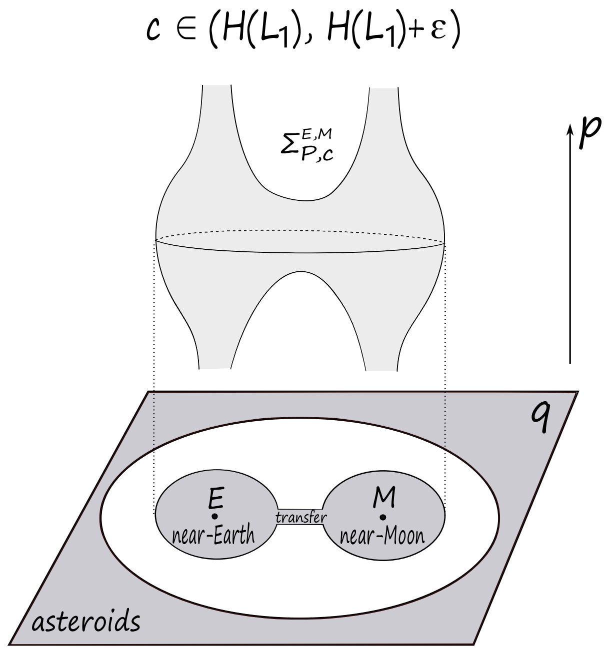


Figure: Morse theory in the three-body problem.

# Moser regularization

$H$ is singular at *collisions* ($q = E$ or $q = M \rightsquigarrow p = \infty$), but can be regularized via Moser's recipe:

$$(q,p) \overset{\text{switch}}{\longmapsto} (-p,q) \overset{\text{stereo. proj.}}{\longmapsto} (\xi,\eta) \in T^*S^3$$

We get compactifications for spatial energy levels:

$$\Sigma_c^E \rightsquigarrow \overline{\Sigma}_c^E \cong S^*S^3.$$

$$\Sigma_c^M \rightsquigarrow \overline{\Sigma}_c^M \cong S^*S^3.$$

$$\Sigma_c^{E,M} \rightsquigarrow \overline{\Sigma}_c^{E,M} \cong S^*S^3 \# S^*S^3.$$

Similarly, the planar problem level sets get compactified to $\overline{\Sigma}_{P,c}^E \cong S^*S^2 = \mathbb{R}P^3$, $\overline{\Sigma}_{P,c}^M \cong S^*S^2 = \mathbb{R}P^3$, $\overline{\Sigma}_{P,c}^{E,M} \cong \mathbb{R}P^3 \# \mathbb{R}P^3$.

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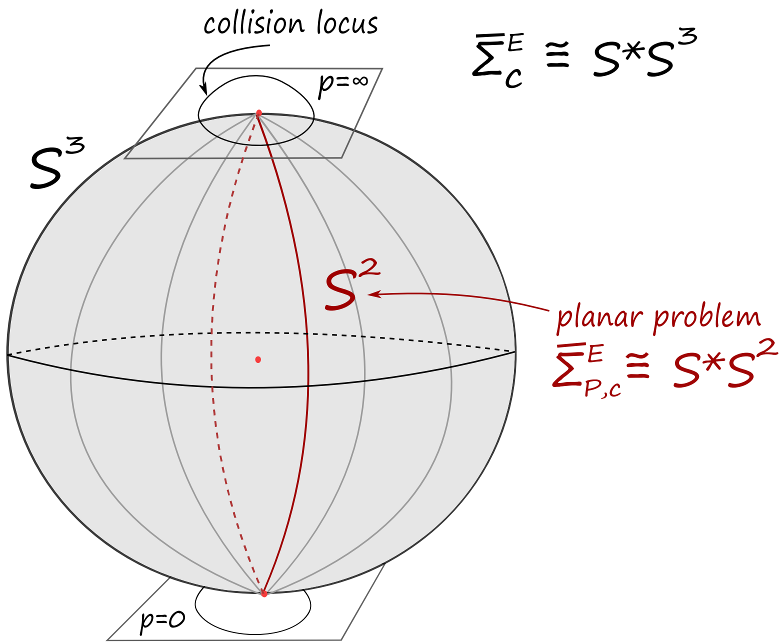


Figure: The Moser-regularized level set near $E$.

# Contact geometry of the three-body problem

**Theorem (planar case: Albers-Frauenfelder-van Koert-Paternain '12, spatial case: Cho-Jung-Kim '19)**

*For $\mu \in (0,1)$, $c < H(L_1)$, $\overline{\Sigma}_c^E$ and $\overline{\Sigma}_c^M$ are contact-type, and so is $\overline{\Sigma}_c^{E,M}$ for $c \in (H(L_1), H(L_1)+\epsilon)$ for some $\epsilon > 0$. As contact manifolds:*

$$\overline{\Sigma}_c^E \cong \overline{\Sigma}_c^M \cong (S^*S^3, \xi_{std}),$$

$$\overline{\Sigma}_c^{E,M} \cong (S^*S^3, \xi_{std})\#(S^*S^3, \xi_{std}).$$

*The planar problem is a flow-invariant codim-2 contact submanifold:*

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# Poincaré-Birkhoff and the planar problem

In his long search for closed orbits in the planar three-body problem, Poincaré's approach can be reduced to:

(1) Finding a global surface of section for the dynamics;

(2) Proving a fixed point theorem for the arising return map.

This is the setting for Poincaré-Birkhoff's theorem:

*An area-preserving homeomorphism of an annulus that rotates the two boundaries in opposite directions (the twist condition) has at least two fixed points.*

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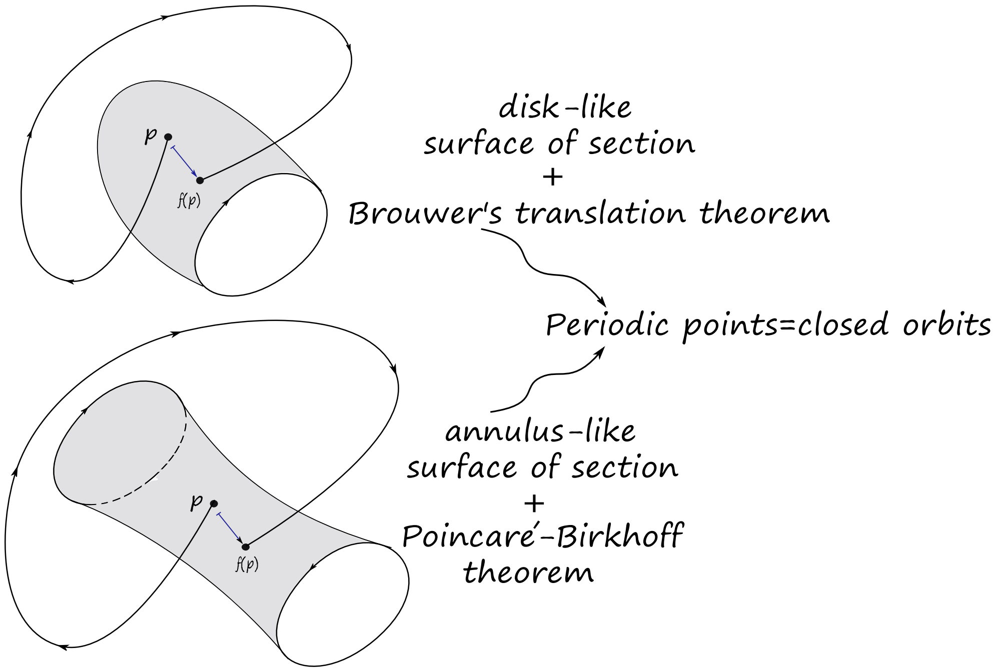


Figure: Obtaining closed orbits.

# Open book decompositions

An *open book decomposition* on a closed odd-dimensional manifold $M$ is a fibration $\pi : M \backslash B \to S^1$, where $B \subset M$ is a closed codimension-2 submanifold with trivial normal bundle, and $\pi(b, r, \theta) = \theta$ on some collar neighbourhood $B \times \mathbb{D}^2$ of $B$.

**Abstract data:** *page* $P = \overline{\pi^{-1}}(pt)$ (with $B = \partial P$ *binding*), *monodromy* $\phi : P \xrightarrow{\cong} P$, $\phi|_B = id$.

$$(P, \phi) \rightsquigarrow M = \mathbf{OB}(P, \phi) = P_\phi \bigcup B \times \mathbb{D}^2,$$

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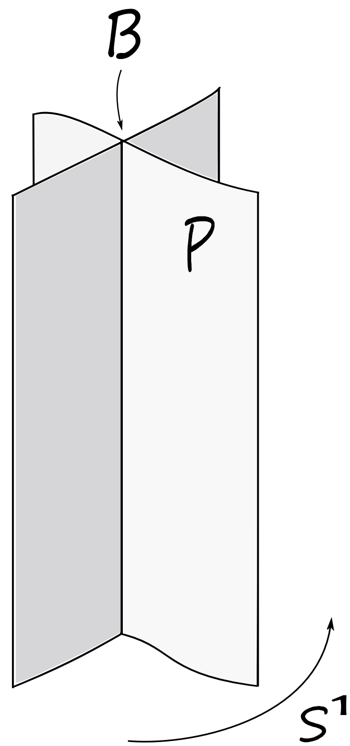

# Global hypersurfaces of section

If $\varphi_t : M \to M$ is a flow on $M$ generated by an autonomous vector field $X$, then $\pi$ is *adapted to the dynamics* if $B$ is $\varphi_t$-invariant (i.e. $X|_B$ is tangent to $B$), and $X$ is transverse to the interior of all pages.

Each page $P$ is a *global hypersurface of section*, i.e. it is codimension-1, $B = \partial P$ is a union of orbits, and the orbits of all points in $M \backslash B$ meet the interior of each page transversely in the future and past.

$\rightsquigarrow$ Poincaré return map $f : \text{int}(P) \to \text{int}(P)$.

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$$\rightsquigarrow \text{Poincaré return map } f : \text{int}(P) \to \text{int}(P).$$

# Open books and surfaces of section in the planar problem: historical remarks

**Planar situation:** smoothly $\mathbb{R}P^3 = \mathbf{OB}(\mathbb{D}^*S^1, \tau_0^2)$, where $\tau_0$ =Dehn twist along $S^1 \subset \mathbb{D}^*S^1$.

**Perturbative methods:**

- If $\mu \sim 0$ is small and $c < H(L_1)$, Poincaré [P12] provides annulus-like global surfaces of section by perturbing the rotating Kepler problem.
- If $c \ll H(L_1)$ and $\mu \in (0,1)$, Conley [C63] shows there are annulus-like surfaces of section and the return map is a Birkhoff twist map, and uses Poincaré-Birkhoff.
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**convexity range:** $\mathcal{C} = \{(\mu, c),\ c < H(L_1) :$ Levi-Civita regularization of planar problem is convex$\}$.

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# Step 1: Open books in the spatial three-body problem

$\overline{\Sigma}_c = H^{-1}(c)$ compact and connected component of a (regularized) energy hypersurface in the SCR3BP.

**Theorem (M.–van Koert)**

*For $\mu \in (0,1)$, we have*

$$\overline{\Sigma}_c = \begin{cases} \mathbf{OB}(\mathbb{D}^* S^2, \tau^2), & \text{if } c < H(L_1) \\ \mathbf{OB}(\mathbb{D}^* S^2 \natural \mathbb{D}^* S^2, \tau_1^2 \circ \tau_2^2), & \text{if } c \in (H(L_1), H(L_1) + \epsilon), \end{cases}$$

*which are adapted to the dynamics. Here, $\tau$ is the Dehn-Seidel twist along the zero section $S^2 \subset \mathbb{D}^* S^2$.*

Binding $B = S^* S^2 = \partial \mathbb{D}^* S^2 = \mathbb{R}P^3 =$ planar problem for energy $c$.

# Step 1: Open books in the spatial three-body problem

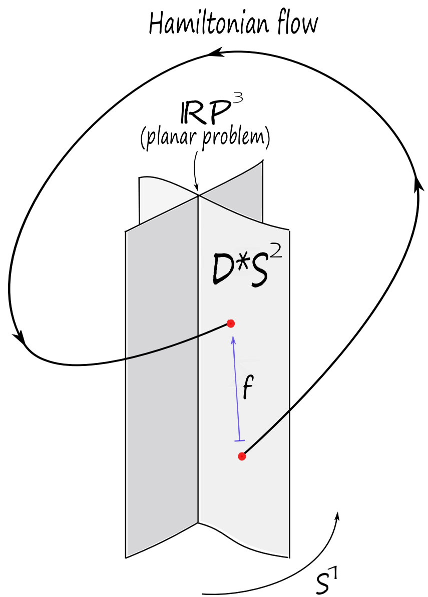


Figure: The open book in the spatial problem for $c < H(L_1)$.

# Basic idea

Let $B = \{p_3 = q_3 = 0\}$ (planar problem). Define

$$\pi(q,p) = \frac{q_3 + ip_3}{\|q_3 + ip_3\|} \in S^1, \; d\pi = \frac{p_3 dq_3 - q_3 dp_3}{p_3^2 + q_3^2}.$$

Then

$$d\pi(X_H) = \frac{p_3^2 + q_3^2 \cdot \left(\frac{1-\mu}{\|q-E\|^3} + \frac{\mu}{\|q-M\|^3}\right)}{p_3^2 + q_3^2} > 0,$$

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**Problem:** This does not extend to the collision locus.

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# Physical interpretation

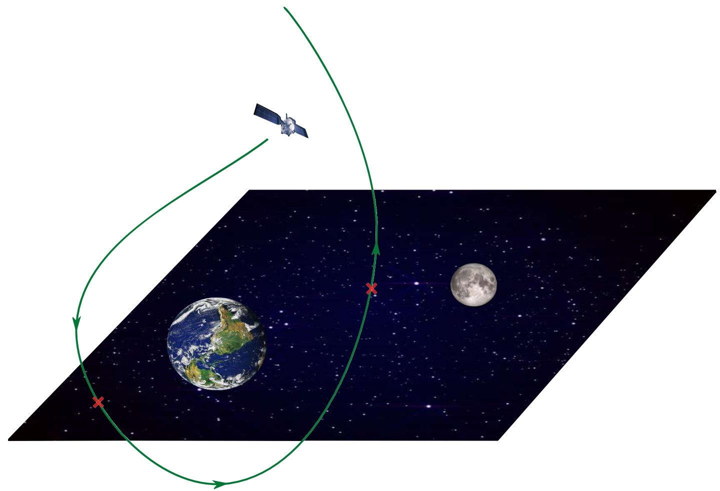

Figure: The $\pi/2$-page corresponds to $q_3 = 0$, $p_3 > 0$, and means that the spatial orbits of $S$ are transverse to the plane spanned by $E, M$ away from collisions.

# Polar orbits

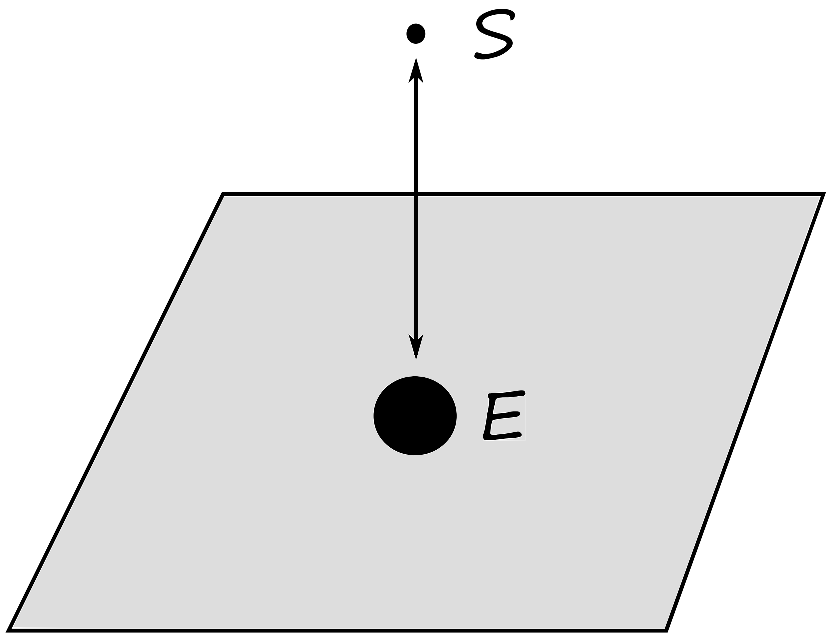


Figure: Polar orbits prevent transversality on the collision locus.

# Return map

**Theorem (M.–van Koert)**

*For every $\mu \in (0, 1]$, $c < H(L_1)$, and page $P$, the return map $f$ extends smoothly to the boundary $B = \partial P$, and in the interior it is an exact symplectomorphism*

$$f = f_{c,\mu} : (int(P), \omega) \to (int(P), \omega),$$

*where $\omega = d\alpha|_P$, $\alpha = \alpha_{\mu,c}$ contact form. Moreover, $f$ is Hamiltonian in the interior, and the Hamiltonian isotopy extends smoothly to the boundary.*

Here, $\omega$ degenerates at $B$, but after a continuous conjugation, it is *deformation equivalent* to the standard symplectic form. The Hamiltonian is *not* rel boundary.

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# Spatial vs Planar orbits

Note that

$$\mathrm{Fix}(f^k) = \mathrm{IntFix}(f^k) \bigcup \mathrm{BdyFix}(f^k),$$

where

$$\mathrm{IntFix}(f^k) \longleftrightarrow \{\text{spatial orbits of period } k\}$$

$$\mathrm{BdyFix}(f^k) \to \{\text{planar orbits}\}$$

**Goal:** Find *interior* periodic points with arbitrary large minimal $k$.

# Step 2: Fixed point theory of Hamiltonian twist maps

$(W, \omega = d\lambda)$ Liouville domain, $\alpha = \lambda|_B$. Let $f : (W, \omega) \to (W, \omega)$ be a Hamiltonian symplectomorphism.

**Definition**

$f$ is a *Hamiltonian twist map* if there exists a time-dependent Hamiltonian $H : \mathbb{R} \times W \to \mathbb{R}$ such that:

- $H$ is *smooth* (or $C^2$);
- $f = \phi_H^1$;
- There exists a smooth function $h : \mathbb{R} \times B \to \mathbb{R}$ which is *positive* and
$$X_{H_t}|_B = h_t R_\alpha.$$

# Fixed-point theorem

**Theorem (M.–van Koert, Generalized Poincaré–Birkhoff theorem)**

*Suppose that $f$ is an exact symplectomorphism of a Liouville domain $(W, \lambda)$, and let $\alpha = \lambda|_B$. Assume the following:*

- ***(Hamiltonian twist map)*** *$f$ is a Hamiltonian twist map;*
- ***(index-definiteness)*** *If $\dim W \geq 4$, then assume $c_1(W)|_{\pi_2(W)} = 0$, and $(\partial W, \alpha)$ is strongly index-definite. In addition, assume all fixed points of $f$ are isolated;*
- ***(Symplectic homology)*** *$SH_\bullet(W)$ is infinite dimensional.*

*Then $f$ has simple interior periodic points of arbitrarily large (integer) period.*

# A few remarks

- Strong index definiteness is a technical assumption, implied by strict convexity.
- If dim $W = 2$, dim $SH_\bullet(W) = \infty$ iff $W \neq \mathbb{D}^2$.
- A very vast generalization of the classical Poincaré-Birkhoff theorem, in the spirit of the Conley conjecture (good).
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# A few remarks

- Strong index definiteness is a technical assumption, implied by strict convexity.
- If $\dim W = 2$, $\dim SH_\bullet(W) = \infty$ iff $W \neq \mathbb{D}^2$.
- A very vast generalization of the classical Poincaré-Birkhoff theorem, in the spirit of the Conley conjecture (good).
- We couldn't check the twist condition in the three-body problem (not so good).

# Holomorphic dynamics

**Observation:** the adapted open book $\mathbf{OB}(\mathbb{D}^*S^2, \tau^2)$ is *iterated planar* (IP), i.e. the page $\mathbb{D}^*S^2 = \mathbf{LF}(\mathbb{D}^*S^1, \tau_P^2)$ admits a Lefschetz fibration with genus zero fibers, all inducing the open book $\mathbf{OB}(\mathbb{D}^*S^1, \tau_P^2)$ at the binding $\mathbb{R}P^3$.

# $T^*S^2 = \mathbf{LF}(T^*S^1, \tau_P^2)$

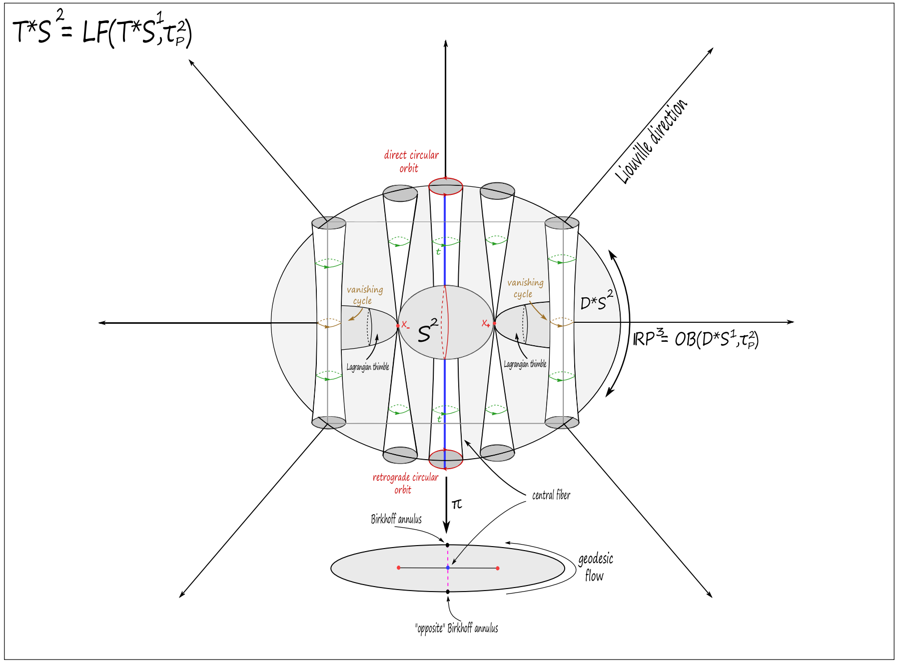


Figure: The standard Lefschetz fibration on $T^*S^2$.

# Abstract page

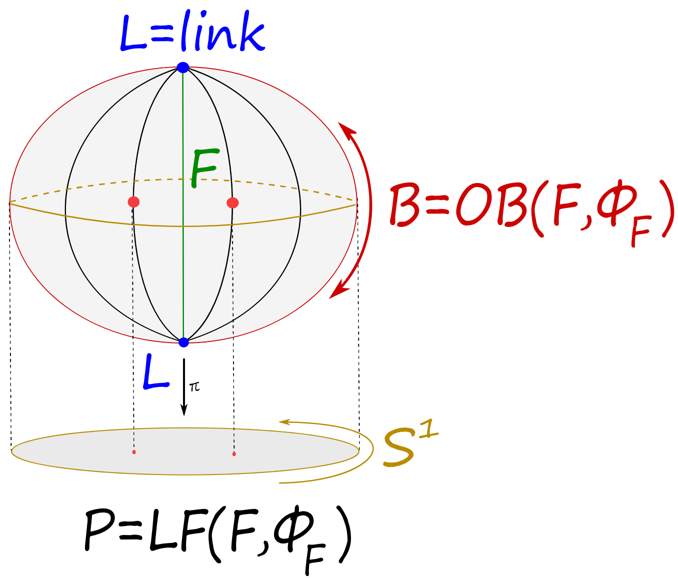


Figure: Abstractly, the compact version of the Lefschetz fibration on a page $P$. $F$ is the regular fiber, $L = \partial F$ is the "binding of the binding" $B$, a link.

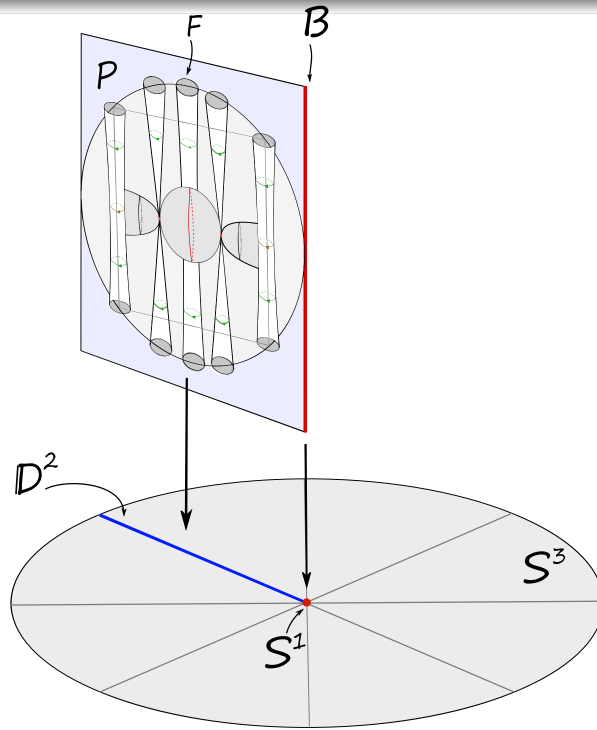


Figure: The moduli space of fibers is a copy of $S^3 = \mathbf{OB}(\mathbb{D}^2, \mathbb{1})$.

# Contact structures and Reeb dynamics on moduli

Let $(M, \xi_M) = \mathbf{OB}(P, \phi)$ be an IP 5-fold, $P = \mathbf{LF}(F, \phi_F)$.

$\mathbf{Reeb}(P, \phi) = \{\alpha$ adapted contact form: $\alpha|_B$ adapted to $B = \mathbf{OB}(F, \phi_F)\}$.

**Theorem (M., Contact structures and Reeb dynamics on moduli)**

*For a given $\alpha \in$ **Reeb**$(P, \phi)$, there is a moduli space $\mathcal{M}$ of $d\alpha$-symplectic copies of F foliating M, forming the fibers of a Lefschetz fibration on each page. $\mathcal{M}$ is a contact manifold $(\mathcal{M}, \xi_{\mathcal{M}}) \cong (S^3, \xi_{std}) = \mathbf{OB}(\mathbb{D}^2, \mathbb{1})$.*

*Any $\alpha \in$ **Reeb**$(P, \phi)$ induces a contact form $\alpha_{\mathcal{M}} \in$ **Reeb**$(\mathbb{D}^2, \mathbb{1})$, $\ker \alpha_{\mathcal{M}} = \xi_{\mathcal{M}}$, adapted to a trivial open book of the form $\theta_{\mathcal{M}} : \mathcal{M} \backslash \mathcal{M}_B \cong S^3 \backslash S^1 \to S^1$.*

# Idea: fiber-wise integration

The contact form $\alpha_{\mathcal{M}}$ is defined via

$$(\alpha_{\mathcal{M}})_u(v) = \int_{z \in F_u} \alpha_z(v(z))dz,$$

where $F_u = \text{im}(u)$, $dz = d\alpha|_{F_u}$, $u \in \mathcal{M}$, $v \in T\mathcal{M}$. Its Reeb vector field $R_{\mathcal{M}}$ is defined via

$$\mathbf{D}_u R_{\mathcal{M}} = 0, \text{ where } \mathbf{D}_u = \text{ linearized CR-operator},$$

$$1 = (\alpha_{\mathcal{M}})_u(R_{\mathcal{M}}(u)) = \int_{z \in F_u} \alpha_z(R_{\mathcal{M}}(z))dz,$$

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# Return map and symplectic tomographies

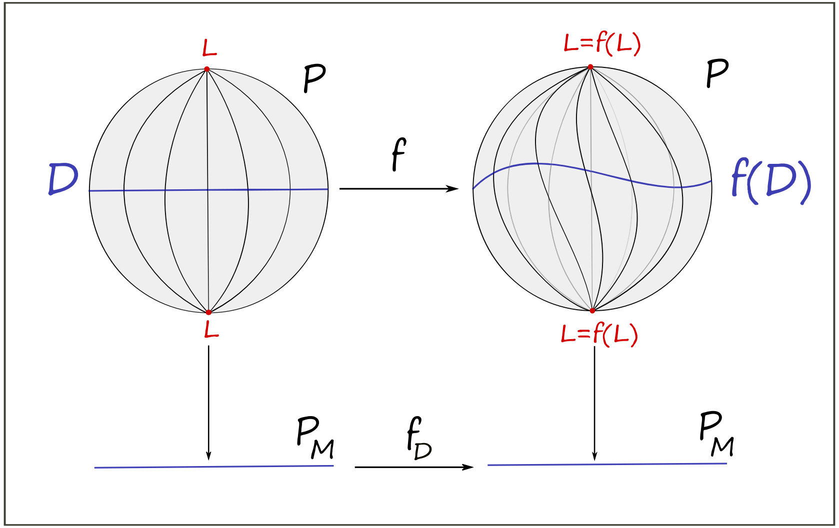


Figure: The return map $f$ might not preserve the symplectic foliation. One can take *symplectic tomographies D* (a symplectic 2-disk) to induce return maps $f_D$ on $\mathcal{M}$.

# Shadowing cone

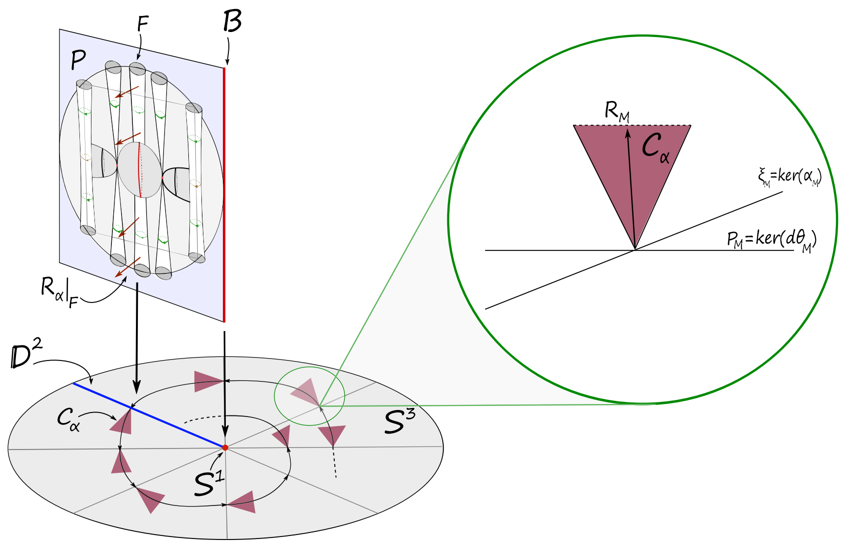


Figure: The shadowing cone is $C_\alpha = \pi_*(\ker d\alpha)$. Orbits of $\alpha$ project to orbits of the cone, which are transverse to $\xi_{\mathcal{M}}$ and to every page. The Reeb vector field $R_{\mathcal{M}}$ spans the average direction of $C_\alpha$.

# Holomorphic shadow

Define the *holomorphic shadow map* as

$$\mathbf{HS}: \mathbf{Reeb}(P, \phi) \to \mathbf{Reeb}(\mathbb{D}^2, \mathbb{1})$$

$$\alpha \mapsto \alpha_{\mathcal{M}}.$$

**Integrable case:** Rotating Kepler problem $\mapsto$ Hopf flow on $S^3$.

The return map preserves the foliation. The two nodal singularities are fixed, and correspond to the polar orbits. The map is a classical twist map on the annuli fibers.

**Theorem (M., Reeb lifting theorem)**

**HS** *is surjective.*

In other words, Reeb dynamics in $M$ is at least as complex as Reeb dynamics in $S^3$.

**New program:** Try to "lift" knowledge from dynamics on $S^3$.

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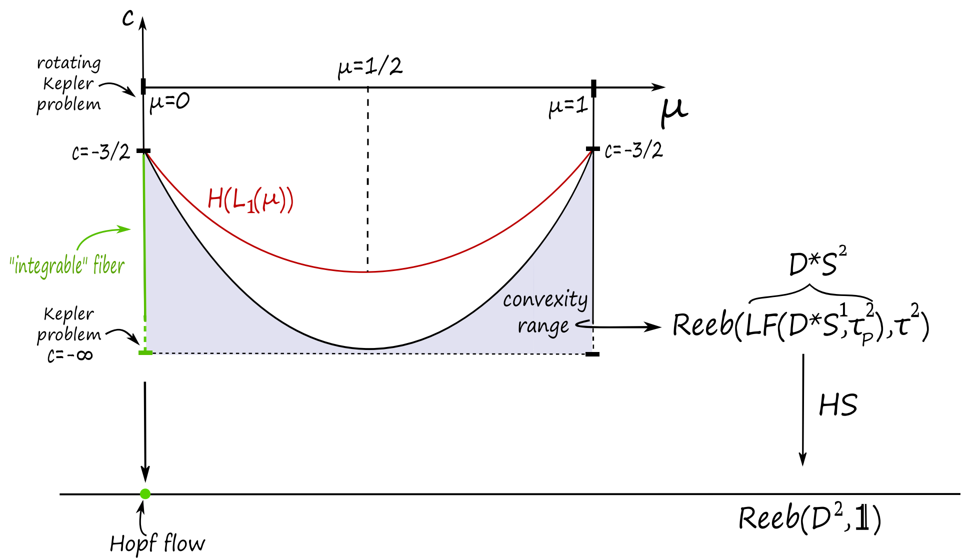
# Case of three-body problem

If $(\mu, c) \in \mathcal{C}$, combining our adapted open book with [HSW] on $B = \mathbb{R}P^3 \rightsquigarrow \alpha_{\mu,c} \in$ **Reeb**$(\mathbb{D}^* S^2, \tau^2)$.



# Further directions: Entropy

Joint work in progress with Umberto Hrynewicz, Abror Pirnapasov:

**Claim 1:** $C^\infty$-generic Reeb flows on any closed 3-fold have positive topological entropy.

Pull back via the shadow map $\rightsquigarrow$

**Claim 2:** $C^\infty$-generic Reeb flows in **Reeb**$(P, \phi)$ also have positive topological entropy, for every IP 5-fold, generated by purely spatial orbits.

# Closing remarks

- Hamiltonian maps which are not the identity at the boundary should perhaps be studied more systematically, specially in higher dimensions.
- The Hamiltonian twist condition, if true at all, seems HARD to check.
  Enter the famous Katok examples! they are a counterxample to the conclusion of the theorem, i.e. they are not twist maps. BUT they are arbitrarily close to the Kepler problem (geodesic flow on $S^3$).
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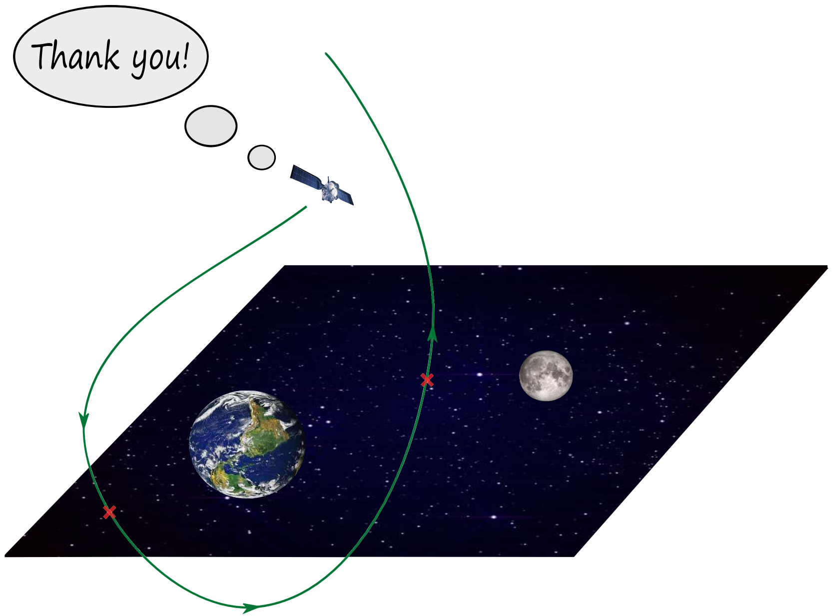

Thank you!

# Complementary slides: Index growth

We call a strict contact manifold $(Y, \xi = \ker \alpha)$ *strongly index-definite* if the contact structure $(\xi, d\alpha)$ admits a symplectic trivialization $\epsilon$ so that:

- There are constants $c > 0$ and $d \in \mathbb{R}$ such that for every Reeb chord $\gamma : [0, T] \to Y$ of Reeb action $T = \int_0^T \gamma^*\alpha$ we have
$$|\mu_{RS}(\gamma; \epsilon)| \geq cT + d,$$
where $\mu_{RS}$ is the Robbin–Salamon index.

Drop absolute value $\rightsquigarrow$ index-*positive*.

# Complementary slides: Examples of index-positivity

**Lemma (Some examples)**

- *If* $(Y, \alpha) \subset \mathbb{R}^4$ *is a strictly convex hypersurface, then it is strongly index-positive.*
- *If* $(Y, \ker \alpha) = (S^*Q, \xi_{std})$ *is symplectically trivial and* $(Q, g)$ *has positive sectional curvature, then* $(Y, \alpha)$ *is strongly index-positive.*

# Complementary slides: special case of fixed-point theorem

**Theorem (M.–van Koert, special case)**

*Let $W \subset (T^*M, \lambda_{can})$ be fiber-wise star-shaped, with $M$ simply connected, orientable and closed. Let $f : W \to W$ be a Hamiltonian twist map. Assume:*

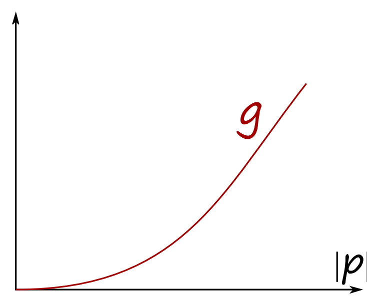
- *Reeb flow on $\partial W$ is strongly index-positive; and*
- *All fixed points of $f$ are isolated.*

*Then $f$ has simple interior periodic points of arbitrarily large period.*

# Complementary slides: Toy example

$Q = S^n$ with round metric.
$H : T^*Q \to \mathbb{R}$, $H(q,p) = 2\pi|p|$ *not* smooth at zero section.
Then $\phi_H^1 = id$, all orbits are periodic with same period.

Let $K = 2\pi g$, with $g = g(|p|)$ smoothing of $|p|$ near $p = 0$. Then $\phi_K^1 = \phi_G^{2\pi g'(|p|)}$, where $\phi_G^t$ geodesic flow, is a Hamiltonian twist map. It has simple orbits of arbitrary period ($g'(|p|) = l/k$ coprime $\rightsquigarrow$ $k$-periodic orbit).



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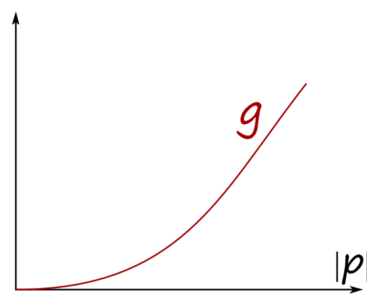

# Complementary slides: dynamical applications

**Definition**

Let $P$ be a page, and $f : \mathrm{int}(P) \to \mathrm{int}(P)$ a return map. A *fiber-wise* $k$-recurrent point is $x \in \mathrm{int}(P)$ such that $f(\mathcal{M}_x) \cap \mathcal{M}_x \neq \emptyset$.

This is a "symplectic version" of a leaf-wise intersection.

**Theorem (M.)**

*In the SCR3BP, for every $k$, one can find sufficently small perturbations of the integrable cases which admit infinitely many fiber-wise $k$-recurrent points.*

# More further directions: Lagrangians

## Conjecture (Long interior chords)

*Suppose that $f$ is an exact symplectomorphism of a Liouville domain $(W, \lambda)$, let $\alpha = \lambda|_B$, and $L \subset (W, \lambda)$ exact, spin, Lagrangian with Legendrian boundary. Assume the following:*

- ***(Hamiltonian twist map)*** *$f$ is a Hamiltonian twist map;*
- ***(index-definiteness)*** *If $\dim W \geq 4$, then assume $c_1(W)|_{\pi_2(W)} = 0$, and $(\partial W, \alpha)$ is strongly index-definite;*
- ***(Wrapped Floer homology)*** *$WFH_\bullet(L)$ is infinite dimensional.*

*Then $f^k(int(L)) \cap int(L)$ is non-empty for $k$ arbitrarily large.*

Motivation: Finding long spatial collision orbits in the 3BP.

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