Super-rigidity and bifurcations of embedded curves in Calabi-Yau 3-folds

Mohan Swaminathan
Princeton University
(Based on joint work with Shaoyun Bai)
June 25, 2021
Overview

1 Background
   - Embedded curves and super-rigidity
   - Wendl’s Theorem
   - BPS invariants and Gopakumar–Vafa formula

2 Results
   - Bifurcations
   - Obstruction bundles
   - An application

3 Further directions
Fix a closed symplectic Calabi–Yau 3-fold \((X, \omega)\), i.e., \(\dim X = 6\) and 
\[c_1(TX, \omega) = 0.\]
Fix a closed symplectic Calabi–Yau 3-fold \((X, \omega)\), i.e., \(\dim X = 6\) and \(c_1(TX, \omega) = 0\).

Given \(J \in \mathcal{J}(X, \omega)\), \(g \geq 0\) and \(A \in H_2(X, \mathbb{Z})\), the moduli space \(\overline{M}_g(X, J, A)\) has virtual dimension 0.
Fix a closed symplectic Calabi–Yau 3-fold \((X, \omega)\), i.e., \(\dim X = 6\) and \(c_1(TX, \omega) = 0\).

Given \(J \in \mathcal{J}(X, \omega)\), \(g \geq 0\) and \(A \in H_2(X, \mathbb{Z})\), the moduli space \(\overline{M}_g(X, J, A)\) has virtual dimension 0.

\(\leadsto\) Gromov–Witten invariant \(GW_{A,g} \in \mathbb{Q}\), independent of \(J\).
Fix a closed symplectic Calabi–Yau 3-fold \((X, \omega)\), i.e., \(\dim X = 6\) and \(c_1(TX, \omega) = 0\).

Given \(J \in \mathcal{J}(X, \omega)\), \(g \geq 0\) and \(A \in H_2(X, \mathbb{Z})\), the moduli space \(\overline{M}_g(X, J, A)\) has virtual dimension 0.

\(\rightsquigarrow\) Gromov–Witten invariant \(GW_{A, g} \in \mathbb{Q}\), independent of \(J\).

Because of multiple covers, these invariants are not \(\mathbb{Z}\)-valued and don’t directly enumerate curves.
Fact

Away from a codimension 2 subset of $\mathcal{J}(X, \omega)$, all simple holomorphic curves are embedded and have pairwise disjoint images.

- Restrict attention to $\mathcal{J}$ as in the above fact.
Fact

Away from a codimension 2 subset of $\mathcal{J}(X, \omega)$, all simple holomorphic curves are embedded and have pairwise disjoint images.

- Restrict attention to $J$ as in the above fact.
- Any non-constant $J$-holomorphic stable map $f' : \Sigma' \to X$ can then be factored uniquely as

$$\Sigma' \xrightarrow{\varphi} \Sigma \xrightarrow{f} X$$

where $\Sigma$ is a smooth closed Riemann surface, $f$ is a $J$-holomorphic embedding and $\varphi$ is holomorphic.
Definition (Super-rigidity)

$J \in \mathcal{J}(X, \omega)$ is called super-rigid if, for all stable $J$-holomorphic maps

$$\Sigma' \xrightarrow{\varphi} \Sigma \subset X$$

we have $\ker(\varphi^* D_{\Sigma, J}^N) = 0$, where $D_{\Sigma, J}^N$ is the normal Cauchy–Riemann operator of the embedded $J$-curve $\Sigma \subset X$. 

Definition (Super-rigidity)

\( J \in \mathcal{J}(X, \omega) \) is called **super-rigid** if, for all stable \( J \)-holomorphic maps \( \Sigma' \xrightarrow{\varphi} \Sigma \subset X \), we have \( \ker(\varphi^* D_{\Sigma,J}^N) = 0 \), where \( D_{\Sigma,J}^N \) is the **normal Cauchy–Riemann operator** of the embedded \( J \)-curve \( \Sigma \subset X \).

- If \( J \) is super-rigid, then given any sequence of embedded \( J_n \)-curves \( \Sigma_n \subset X \) (of bounded genus and area), with \( J_n \to J \), we can find a subsequence converging to an embedded \( J \)-curve \( \Sigma \subset X \).
Definition (Super-rigidity)

\( J \in \mathcal{J}(X, \omega) \) is called super-rigid if, for all stable \( J \)-holomorphic maps

\[
\Sigma' \xrightarrow{\varphi} \Sigma \subset X
\]

we have \( \ker(\varphi^* D_{\Sigma,J}^N) = 0 \), where \( D_{\Sigma,J}^N \) is the normal Cauchy–Riemann operator of the embedded \( J \)-curve \( \Sigma \subset X \).

- If \( J \) is super-rigid, then given any sequence of embedded \( J_n \)-curves \( \Sigma_n \subset X \) (of bounded genus and area), with \( J_n \rightarrow J \), we can find a subsequence converging to an embedded \( J \)-curve \( \Sigma \subset X \).
- This allows us to separate embedded curves from multiple covers!
Theorem (Wendl 2019, arXiv:1609.09867)

The subset of $\mathcal{J}(X, \omega)$ where super-rigidity fails has codimension $\geq 1$. In particular, the generic $J$ is super-rigid.

To show symplectic invariance, we must investigate what happens when we cross the codimension 1 strata ("walls").
Theorem (Wendl 2019, arXiv:1609.09867)

The subset of $\mathcal{J}(X,\omega)$ where super-rigidity fails has codimension $\geq 1$. In particular, the generic $J$ is super-rigid.

- The actual result determines the codimensions of the various strata of this subset (corresponding to the Galois group of the covers involved and their representations).
Theorem (Wendl 2019, arXiv:1609.09867)

The subset of $\mathcal{J}(X,\omega)$ where super-rigidity fails has codimension $\geq 1$. In particular, the generic $J$ is super-rigid.

- The actual result determines the codimensions of the various strata of this subset (corresponding to the Galois group of the covers involved and their representations).
- This provides a strategy to define $\mathbb{Z}$-valued counts of embedded curves using super-rigid $J$. 
Wendl’s Theorem

Theorem (Wendl 2019, arXiv:1609.09867)

The subset of $J(X, \omega)$ where super-rigidity fails has codimension $\geq 1$. In particular, the generic $J$ is super-rigid.

- The actual result determines the codimensions of the various strata of this subset (corresponding to the Galois group of the covers involved and their representations).
- This provides a strategy to define $\mathbb{Z}$-valued counts of embedded curves using super-rigid $J$.
- To show symplectic invariance, we must investigate what happens when we cross the codimension 1 strata (“walls”).
Conjecture (Gopakumar–Vafa ’98)

There exist integers $BPS_{A,h}$ for all $h \geq 0$ and $A \in H_2(X, \mathbb{Z})$ satisfying the following identity

$$
\sum_{A \neq 0, g \geq 0} GW_{A,g} t^{2g-2} q^A = \sum_{A \neq 0, h \geq 0} BPS_{A,h} \sum_{k=1}^{\infty} \frac{1}{k} \left( 2 \sin \left( \frac{kt}{2} \right) \right)^{2h-2} q^{kA}
$$
**Conjecture (Gopakumar–Vafa ’98)**

There exist integers $BPS_{A,h}$ for all $h \geq 0$ and $A \in H_2(X, \mathbb{Z})$ satisfying the following identity

$$
\sum_{A \neq 0, g \geq 0} GW_{A,g} t^{2g-2} q^A = \sum_{A \neq 0, h \geq 0} BPS_{A,h} \sum_{k=1}^{\infty} \frac{1}{k} \left(2 \sin \left(\frac{kt}{2}\right)\right)^{2h-2} q^{kA}
$$

**Theorem (Ionel–Parker, 2018)**

There exist integers $BPS_{A,h}$ for $h \geq 0$ and $A \in H_2(X, \mathbb{Z})$ satisfying the Gopakumar–Vafa formula.
Recently, Doan–Ionel–Walpuski (arXiv:2103.08221) have also shown that for any $A \in H_2(X, \mathbb{Z})$, we have $\text{BPS}_{A,h} = 0$ for $h \gg 0$. However, neither of these proofs show how to interpret the integers $\text{BPS}_{A,h}$ enumeratively.

Motivating question
How to define $\mathbb{Z}$-valued symplectic invariants by counting embedded curves? How are these counts related to the BPS invariants?
Recently, Doan–Ionel–Walpuski (arXiv:2103.08221) have also shown that for any $A \in H_2(X, \mathbb{Z})$, we have $\text{BPS}_{A,h} = 0$ for $h \gg 0$.

However, neither of these proofs show how to interpret the integers $\text{BPS}_{A,h}$ enumeratively.
Recently, Doan–Ionel–Walpuski (arXiv:2103.08221) have also shown that for any $A \in H_2(X, \mathbb{Z})$, we have $BPS_{A,h} = 0$ for $h \gg 0$.

However, neither of these proofs show how to interpret the integers $BPS_{A,h}$ enumeratively.

Motivating question:

How to define $\mathbb{Z}$-valued symplectic invariants by counting embedded curves? How are these counts related to the BPS invariants?
Our recent paper (arXiv:2106.01206) addresses parts of this question. For the first question, we study the bifurcations in the space of embedded curves which occur when we cross one of the walls from Wendl’s theorem. For the second question, we study how the (Euler numbers of) obstruction bundles change under some simple bifurcations.
Theorem A (Bai–S., 2021)

Let \( \{J_t\}_{t \in [-1,1]} \) be a generic path in \( \mathcal{J}(X, \omega) \). Assume that there exists an embedded rigid \( J_0 \)-curve \( \Sigma \subset X \) along with a \( d \)-fold genus \( h \) branched multiple cover \( \varphi : \Sigma' \to \Sigma \) which has non-trivial normal deformations. If this cover determines an elementary wall type, then \( \text{Aut}(\varphi) \subset \mathbb{Z}/2\mathbb{Z} \) and the change in the signed count of embedded curves of genus \( h \) and class \( d[\Sigma] \) near \( \varphi \) is given by \( \pm 2/|\text{Aut}(\varphi)| \).
Theorem A (Bai–S., 2021)

Let \( \{J_t\}_{t \in [-1,1]} \) be a generic path in \( \mathcal{J}(X, \omega) \). Assume that there exists an embedded rigid \( J_0 \)-curve \( \Sigma \subset X \) along with a \( d \)-fold genus \( h \) branched multiple cover \( \varphi : \Sigma' \to \Sigma \) which has non-trivial normal deformations. If this cover determines an elementary wall type, then \( \text{Aut}(\varphi) \subset \mathbb{Z}/2\mathbb{Z} \) and the change in the signed count of embedded curves of genus \( h \) and class \( d[\Sigma] \) near \( \varphi \) is given by \( \pm 2/|\text{Aut}(\varphi)| \).

- The technical condition of “elementary wall type” is satisfied by a large class of branched covers. For example, this includes all \( d \)-fold covers \( \Sigma' \to \Sigma \) with generalized automorphism group \( S_d \).
The key ideas already appear in the bifurcation analysis used to define Taubes’ Gromov invariant (‘96). Using Wendl’s theorem, we are able extend these ideas to our case.
The key ideas already appear in the bifurcation analysis used to define Taubes’ Gromov invariant (’96). Using Wendl’s theorem, we are able extend these ideas to our case.

For the proof, we study the local structure near \((J_0, \varphi : \Sigma' \to \Sigma \subset X)\) of the moduli space

\[
\overline{M}_h(X, \{J_t\}, dA)
\]

where \(A = [\Sigma] \in H_2(X, \mathbb{Z})\).
The key ideas already appear in the bifurcation analysis used to define Taubes’ Gromov invariant (’96). Using Wendl’s theorem, we are able extend these ideas to our case.

For the proof, we study the local structure near \((J_0, \varphi : \Sigma' \to \Sigma \subset X)\) of the moduli space

\[
\overline{\mathcal{M}}_h(X, \{J_t\}, dA)
\]

where \(A = [\Sigma] \in H_2(X, \mathbb{Z}).\)

We obtain a local Kuranishi model by applying the implicit function theorem. We then analyze the first few terms in the Taylor expansion of the Kuranishi map to complete the proof.
(z_1, \ldots, z_r) are coordinates on $T_{\varphi} M_h(\Sigma, d)$, $\epsilon$ is a coordinate $\ker(\varphi^* D^N_{\Sigma, J})$ and $Z, Z'$ are the local irreducible components of the moduli space.
Obstruction bundles (I)

Fix a compact Riemann surface $\Sigma$ of genus $g$ and a $\mathbb{C}$-vector bundle $N \to \Sigma$ of rank 2 with $\deg(N) = 2g - 2$. 
Fix a compact Riemann surface $\Sigma$ of genus $g$ and a $\mathbb{C}$-vector bundle $N \to \Sigma$ of rank 2 with $\deg(N) = 2g - 2$.

**Definition**

A Cauchy–Riemann operator $D$ on $N$ is said to be **super-rigid**, if $\ker(\varphi^* D) = 0$ for all (possibly branched) holomorphic covers $\varphi : \Sigma' \to \Sigma$. For super-rigid $D$ and integers $d \geq 2$ and $h \geq 0$, we define the (canonically oriented) **cokernel bundle** $N_{\Sigma, D}^{(d, h)} \to \overline{\mathcal{M}}_h(\Sigma, d)$ by

$$[\varphi : \Sigma' \to \Sigma] \mapsto \text{coker}(\varphi^* D).$$

Since, $\text{vdim } \overline{\mathcal{M}}_h(\Sigma, d) = \text{rank } N_{\Sigma, D}^{(d, h)}$, this bundle has a well-defined **virtual Euler number** $e_{d, h}(D) \in \mathbb{Q}$. 

Mohan Swaminathan (Princeton)

Bifurcations of embedded curves in CY3’s

June 25, 2021
Theorem B (Bai–S., 2021)

Let $\mathcal{D} = \{D_t\}_{t \in [-1, 1]}$ be a generic 1-parameter family of Cauchy–Riemann operators on $N$. Assume that $([\varphi : \Sigma' \to \Sigma], t) \mapsto \text{coker}(\varphi^* D_t)$ gives a vector bundle of the expected rank on the space

$$\overline{M}_h(\Sigma, d) \times [-1, 1] \setminus \Delta \times \{0\}$$

where $\Delta \subset M_h(\Sigma, d)$ is a finite set where super-rigidity fails for $D_0$. Then,

$$e_{d,h}(D_+) - e_{d,h}(D_-) = \sum_{p \in \Delta} \frac{2 \cdot \text{sgn}(\mathcal{D}, p)}{|\text{Aut}(p)|}$$

with $\text{sgn}(\mathcal{D}, p) \in \{-1, +1\}$ determined by the behavior of $\mathcal{D}$ near $p$. 
Theorem B (Bai–S., 2021)

Let \( D = \{ D_t \}_{t \in [-1,1]} \) be a generic 1-parameter family of Cauchy–Riemann operators on \( N \). Assume that \( ([\varphi : \Sigma' \to \Sigma], t) \mapsto \text{coker}(\varphi^* D_t) \) gives a vector bundle of the expected rank on the space

\[
\overline{M}_h(\Sigma, d) \times [-1, 1] \setminus \Delta \times \{0\}
\]

where \( \Delta \subset M_h(\Sigma, d) \) is a finite set where super-rigidity fails for \( D_0 \). Then,

\[
ed_{d, h}(D_+) - ed_{d, h}(D_-) = \sum_{p \in \Delta} \frac{2 \cdot \text{sgn}(D, p)}{|\text{Aut}(p)|}
\]

with \( \text{sgn}(D, p) \in \{-1, +1\} \) determined by the behavior of \( D \) near \( p \).

The proof is by local finite dimensional reduction to a model case.
### Theorem C (Bai–S., 2021)

Given any primitive homology class \( A \in H_2(X, \mathbb{Z}) \), the number \( BPS_{2A,0}(X) \in \mathbb{Z} \) is a weighted count of embedded \( J \)-holomorphic genus 0 curves (of classes \( 2A \) and \( A \)) when \( J \) is super-rigid.
An application

Theorem C (Bai–S., 2021)

Given any primitive homology class $A \in H_2(X, \mathbb{Z})$, the number $BPS_{2A,0}(X) \in \mathbb{Z}$ is a weighted count of embedded $J$-holomorphic genus 0 curves (of classes $2A$ and $A$) when $J$ is super-rigid.

To define $BPS_{2A,0}$ directly, we count the $2A$ curves $\Sigma'$ with the usual signs, while we count any $A$ curves $\Sigma$ with a weight which counts the signed number of wall crossings along generic path from $D^N_{\Sigma,J}$ to the standard Cauchy–Riemann operator $\bar{\partial}$ on $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. 
An application

**Theorem C (Bai–S., 2021)**

Given any primitive homology class \( A \in H_2(X, \mathbb{Z}) \), the number \( BPS_{2A,0}(X) \in \mathbb{Z} \) is a weighted count of embedded \( J \)-holomorphic genus 0 curves (of classes \( 2A \) and \( A \)) when \( J \) is super-rigid.

- To define \( BPS_{2A,0} \) directly, we count the \( 2A \) curves \( \Sigma' \) with the usual signs, while we count any \( A \) curves \( \Sigma \) with a weight which counts the signed number of wall crossings along generic path from \( D_{\Sigma,J}^N \) to the standard Cauchy–Riemann operator \( \bar{\partial} \) on \( \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \).

- Symplectic invariance of this definition follows from Theorem A.
Theorem C (Bai–S., 2021)

Given any primitive homology class $A \in H_2(X, \mathbb{Z})$, the number $BPS_{2A,0}(X) \in \mathbb{Z}$ is a weighted count of embedded $J$-holomorphic genus 0 curves (of classes $2A$ and $A$) when $J$ is super-rigid.

- To define $BPS_{2A,0}$ directly, we count the $2A$ curves $\Sigma'$ with the usual signs, while we count any $A$ curves $\Sigma$ with a weight which counts the signed number of wall crossings along generic path from $D^N_{\Sigma,J}$ to the standard Cauchy–Riemann operator $\bar{\partial}$ on $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.
- Symplectic invariance of this definition follows from Theorem A.
- The verification of the GV formula uses Theorem B and the standard computation of $e_{2,0}(\bar{\partial})$ for $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$. 
Further directions

We do not give a complete answer to the motivating question due to the following obstacle: our bifurcation analysis only deals with “elementary wall types”. We need to extend it to include other cases such as non-minimal covers, branched covers where not all branch points are distinct, nodal covers (possibly with ghost components). We hope to address this in future work.
We do not give a complete answer to the motivating question due to the following obstacle: our bifurcation analysis only deals with “elementary wall types”. We need to extend it to include other cases such as

- non-minimal covers,
Further directions

We do not give a complete answer to the motivating question due to the following obstacle: our bifurcation analysis only deals with “elementary wall types”. We need to extend it to include other cases such as

- non-minimal covers,
- branched covers where not all branch points are distinct,
Further directions

We do not give a complete answer to the motivating question due to the following obstacle: our bifurcation analysis only deals with “elementary wall types”. We need to extend it to include other cases such as

- non-minimal covers,
- branched covers where not all branch points are distinct,
- nodal covers (possibly with ghost components).

We hope to address this in future work.
Thank you!