Lattice formulas for rational SFT capacities

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Main reference

- J. Chaidez & B. Wormleighton. (2021). *Lattice formulas for rational SFT capacities.* arXiv preprint arXiv:2106.07920.

Symplectic embedding problems

- We are interested in *symplectic embeddings*:

$$\iota\colon (X,\omega) \to (X',\omega') \text{ with } \iota^*\omega' = \omega$$

- In particular, we'll focus on their *obstructive* aspects: defining and studying invariants that obstruct embeddings.
- A *symplectic capacity* is an assignment of a real number $\mathfrak{c}(X,\omega)$ to each symplectic manifold (in some class) such that

$$(X,\omega) \overset{s}{\hookrightarrow} (X',\omega') \Longrightarrow \mathfrak{c}(X,\omega) \le \mathfrak{c}(X',\omega')$$

RSFT capacities

- Siegel recently built a collection of capacities using rational SFT.
- These capacities $\mathfrak{r}_P(X,\omega)$ are indexed by *tangency constraints* for curves (e.g. $P =$ "meet p_1 and meet p_2 with multiplicity 3 relative to a local divisor").
- Very roughly speaking,

$$\mathfrak{r}_P(X,\omega) = \min_C \left\{ \int_C \omega : \; C \text{ satisfies } P \right\}$$

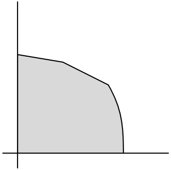
where C ranges over a suitable set of *rational* curves.

Toric domains

- Let $\mu\colon \mathbb{C}^n \to \mathbb{R}^n$ be the moment map for the standard torus action on $\mathbb{C}^n$. Define a *toric domain*

$$X_\Omega := \mu^{-1}(\Omega) \qquad \text{for } \Omega \subseteq \mathbb{R}^n$$

- X_Ω is *strongly convex* if Ω looks like...



- e.g. balls, ellipsoids, polydisks,...

Toric domains

- There are combinatorial formulas for various capacities of toric domains (ECH, Gutt–Hutchings,...) that make computations very accessible.
- These capacities and their combinatorial formulations have helpful interpretations via toric algebraic geometry.
- **We will analyse the RSFT capacities of toric domains through this lens (with some differences).**

Main results

- Throughout we will assume that P is a *lax* tangency constraint: the incidence conditions in P apply to distinct points.
- e.g. $P =$ "meet p_1 and meet p_2 with multiplicity 3" is lax if and only if $p_1 \neq p_2$.

Main results

Theorem (Chaidez–W. '21)

Let X_Ω *be a strongly convex* 4*-d toric domain. There exist combinatorial quantites* $\mathfrak{l}_k(\Omega)$ *and* $\mathfrak{u}_k(\Omega)$ *such that*

$$\mathfrak{l}_k(\Omega) \leq \mathfrak{r}_P(X_\Omega) \leq \mathfrak{u}_k(\Omega)$$

when $\mathrm{codim}(P) = 2k$*.*

Main results

Set $a(\Omega) = \max\{a : (a, 0) \in \Omega\}$ and $b(\Omega) = \max\{b : (0, b) \in \Omega\}$.

Theorem (Chaidez–W. '21)

Let X_Ω be a strongly convex 4-d toric domain with $a(\Omega) = b(\Omega)$. Then

$$\mathfrak{l}_k(\Omega) = \mathfrak{r}_P(X_\Omega) = \mathfrak{u}_k(\Omega)$$

when $\operatorname{codim}(P) = 2k$.

Main results

To a rational-sloped polytope Ω we can associate an algebraic toric surface Y_Ω with an ample divisor A_Ω.

Theorem (Chaidez–W. '21)

Let Ω be a strongly convex rational-sloped polygon. If X is a star-shaped domain that symplectically embeds into Y_Ω then

$$\mathfrak{r}_P(X) \leq \mathfrak{u}_k(\Omega)$$

when $\operatorname{codim}(P) = 2k$.

Main results

Consider the special case $P(k) =$ "meet p with multiplicity $k+1$".

Theorem (Chaidez–W. '21)

Let Ω be a rational-sloped strongly convex polygon. If X is a star-shaped domain that symplectically embeds into $Y_\Omega \times Z$ for some closed Z then

$$\mathfrak{r}_{P(k)}(X) \leq \mathfrak{u}_{k+1}(\Omega)$$

The results above also allow us to completely describe the asymptotics of $\mathfrak{r}_P(X_\Omega)$ as $\operatorname{codim}(P) \to \infty$.

$\mathfrak{l}_k$ and $\mathfrak{u}_k$

- For convex $\Omega \subseteq \mathbb{R}^n$ define the Ω-*norm*

$$\|v\|_\Omega^* := \sup_{u \in \Omega} u \cdot v$$

- Define

$$\mathfrak{l}_k(\Omega) := \min_{k' \geq k} \Big\{ \sum_{i=0}^{k'} \|v_i\|_\Omega^* : \sum_{i=0}^{k'} v_i = 0 \Big\}$$

- This comes from direct arguments in RSFT, and lower bounds $\mathfrak{r}_P(X_\Omega)$ in all dimensions.

$\mathfrak{l}_k$ and $\mathfrak{u}_k$

- Consider the condition on a sequence $v_0, \ldots, v_{k'} \in \mathbb{Z}^2$

(†) if $k' > 2$ and $v_p = e_i$ then there exists $v_q \neq \pm e_i$

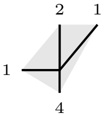
- Define

$$\mathfrak{u}_k(\Omega) := \min_{k' \geq k} \left\{ \sum_{i=0}^{k'} \|v_i\|_\Omega^* : \sum_{i=0}^{k'} v_i = 0 \text{ and } (\dagger) \right\}$$

- This comes from *algebraic geometry*.

Schematic

$\mathrm{GW}_{Y,P}(C) > 0 \Rightarrow \mathfrak{r}_P(X) \leq \int_C \omega$



$C \longleftrightarrow \sum m_i v_i = \mathbf{0}$

$A_\Omega \cdot C = \sum m_i \|v_i\|_\Omega^*$

$-K \cdot C = \sum m_i$

GW invariants with tangency

McDuff–Siegel

immersed PSI spheres with enough moduli

curves in toric varieties

strongly convex

generators for $\mathrm{Eff}(Y_\Omega)^\vee$

needs $(\dagger)$

plumbing

resolving

closure of $\mathrm{im}(\mathbb{C}^\times \to (\mathbb{C}^\times)^2) \subseteq Y_\Omega$

$\mathfrak{r}_P(X_\Omega) \leq \mathfrak{u}_k(\Omega)$

Asymptotics

The bounds $\mathfrak{l}_k$ and $\mathfrak{u}_k$ turn out to have the same asymptotics, giving...

Theorem (Chaidez–W. '21)

Let $\Omega \subseteq \mathbb{R}^2$ *be a strongly convex domain and let* P_i *be a sequence of lax tangency constraints with* $\operatorname{codim}(P_i) \to \infty$. *Then,*

$$\lim_{i\to\infty} \frac{\mathfrak{r}_{P_i}(X_\Omega)}{\operatorname{codim}(P_i)} = \frac{1}{2} \cdot \min_{\mathbf{0}\neq(w_1,w_2)\in\mathbb{Z}^2_{\geq 0}} \frac{\|(w_1,w_2)\|^*_\Omega}{1+w_1+w_2}$$

What is this limit capacity?

Gromov–Witten invariants

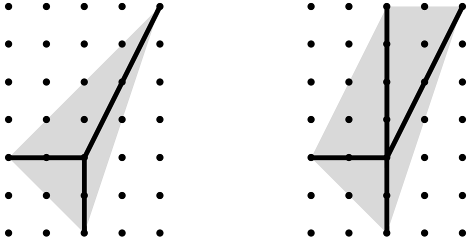
- Let (Y, ω) be a closed symplectic 4-manifold. McDuff–Siegel define a Gromov–Witten invariant $\mathrm{GW}_{Y,P}$ with the property:

$$X \text{ star-shaped and } X \overset{\mathsf{s}}{\hookrightarrow} Y \Longrightarrow \mathfrak{r}_P(X) \leq \int_C \omega$$

  for every curve class C with $\mathrm{GW}_{Y,P}(C) \neq 0$.
- We can (essentially) embed X_Ω in Y_Ω and so Gromov–Witten nonzero curves in Y_Ω give upper bounds on $\mathfrak{r}_P(X_\Omega)$.
- McDuff–Siegel show that if C can be represented by an immersed sphere with positive self-intersections satisfying P and $\mathrm{ind}(C) = \mathrm{codim}(P)$, then $\mathrm{GW}_{Y,P}(C) \neq 0$.

Curves in toric varieties

- A toric variety Y can be encoded by a *fan* Σ.



- Denote the set of rays in Σ by $\Sigma(1)$.
- If $Y = Y_\Omega$ arises from a polytope then Σ is the 'normal fan' of Ω and elements of $\Sigma(1)$ correspond to facet normals.

Curves in toric varieties

- There is a (dual) excision sequence

$$0 \to N_1(Y) \to \mathbb{Z}^{\Sigma(1)} \to \mathbb{Z}^n \to 0$$

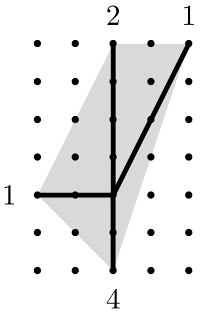
realising curve classes as relations between ray generators.
- A curve class C is *movable* if $C \cdot D \geq 0$ for all effective divisors D.
- Write $\mathrm{Mov}(Y)$ for the cone of movable curves on Y.

Curves in toric varieties

Example

There is a curve class C on $\mathbb{F}_2$ corresponding to

$$1 \cdot (1,2) + 2 \cdot (0,1) + 1 \cdot (-1,0) + 4 \cdot (0,-1) = (0,0)$$



Curves in toric varieties

Example

If Ω is a polytope defining $\mathbb{F}_2$ we can compute $A_\Omega \cdot C =$

$$1 \cdot \|(1,2)\|_\Omega^* + 2 \cdot \|(0,1)\|_\Omega^* + 1 \cdot \|(-1,0)\|_\Omega^* + 4 \cdot \|(0,-1)\|_\Omega^*$$

and $-K_{Y_\Omega} \cdot C =$

$$1 + 2 + 1 + 4$$

This curve class is *movable* because each of the coefficients in the corresponding relation are positive.

$\mathfrak{l}_k$ algebraically

Lemma

$$\mathfrak{l}_k(\Omega) = \min_{C \in \mathrm{Mov}(Y)} \left\{ A_\Omega \cdot C : -K_Y \cdot C \geq k+1 \right\}$$

- The condition $-K_Y \cdot C \geq k+1$ means that C should have enough moduli to find a representative satisfying a tangency constraint P of codimension $2k$.

$\mathfrak{u}_k$ algebraically

- The issue is that not all movable curves with $-K_Y \cdot C \geq k+1$ have $\mathrm{GW}_{Y,P}(C) \neq 0$.
- We find explicit generators of $\mathrm{Mov}(Y_\Omega)$ when Ω is strongly convex – *cocharacter curves* – and apply McDuff–Siegel to conclude that all curve classes **that satisfy** $(\dagger)$ have

$$\mathrm{GW}_{Y_\Omega,P}(C) \neq 0 \text{ when } -K_{Y_\Omega} \cdot C \geq k+1$$

- We deduce

$$\mathfrak{r}_P(X_\Omega) \leq \mathfrak{u}_k(\Omega)$$

$\mathfrak{u}_k$ algebraically (more detail)

- We find generators C_ρ for $\mathrm{Mov}(Y_\Omega)$ indexed by rays $\rho \in \Sigma(1)$.
- Each C_ρ is representable by a rational curve with mild singularities (closure of a cocharacter $\mathbb{C}^\times \to (\mathbb{C}^\times)^2$).
- Thus we can represent any movable curve class as a union of curves in the classes C_ρ. When $(\dagger)$ is satisfied these curves can be resolved and plumbed to produce a *single immersed rational curve with positive self-intersections*.
- This allows us to apply McDuff–Siegel's result to find that all such curve classes have

$$\mathrm{GW}_{Y_\Omega,P}(C) \neq 0 \text{ when } -K_{Y_\Omega} \cdot C \geq k+1$$

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References

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