Symplectically knotted cubes

joint with

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The problem

Take a "good" compact set $K \subset (\mathbb{R}^{2n}, \omega_0)$, $\omega_0 = \sum_{j=1}^n dx_j \wedge dy_j$

like a ball or a cube of an ellipsoid or a polydisc

(M^{2n}, ω) connected symplectic manifold

$\mathrm{Emb}_\omega(K, M)$:= space of symplectic embeddings $K \to M$

Question 1 Is $\mathrm{Emb}_\omega(K, M)$ non-empty ?

Question 2 Is $\mathrm{Emb}_\omega(K, M)$ connected ?

If not, what is $\pi_0(\mathrm{Emb}_\omega(K, M))$?

Question 3 What is the topology (π_k, H_k) of $\mathrm{Emb}_\omega(K, M)$?

Most known results are on Question 1.

Examples

- Gromov's nonsqueezing theorem:

$$B^{2n}(a) \overset{s}{\hookrightarrow} B^2(A) \times \mathbb{R}^{2n-2} \text{ only if } a \leq A$$

- The ball packing problem $\coprod_k B^4(1) \overset{s}{\hookrightarrow} B^4(A)$

  (**Gromov, McDuff–Polterovich, Biran**)

- The problem $E(1, a) \overset{s}{\hookrightarrow} B^4(A)$ (**McDuff–S**)

- Many new recent results, but also many open problems

Question 2: Much less is known ...

Question 3: Almost nothing is known

See however **Anjos–Lalonde–Pinsonnault**

and the recent **Chaidez–Munteanu**

Precisions on Question 2:

$K \stackrel{s}{\hookrightarrow} (M, \omega)$ means:

There exists a symplectic embedding of a neighbourhood of K

Several equivalence relations on $\mathrm{Emb}_\omega(K, M)$ make sense:

$\varphi_1 \sim_0 \varphi_2 \quad :=$

$\exists$ a smooth path $\varphi_t \colon K \overset{s}{\hookrightarrow} M$ connecting φ_1 with φ_2

For K simply connected (as always here) this is the same as:

$\exists$ a Hamiltonian isotopy ϕ_H of M such that

$$\phi_H \circ \varphi_1 \; = \; \varphi_2$$

$\varphi_1 \sim_\omega \varphi_2 \; :=$

$\exists$ a compactly supported symplectomorphism ψ of (M, ω) s.t.

$$\psi \circ \varphi_1 \; = \; \varphi_2$$

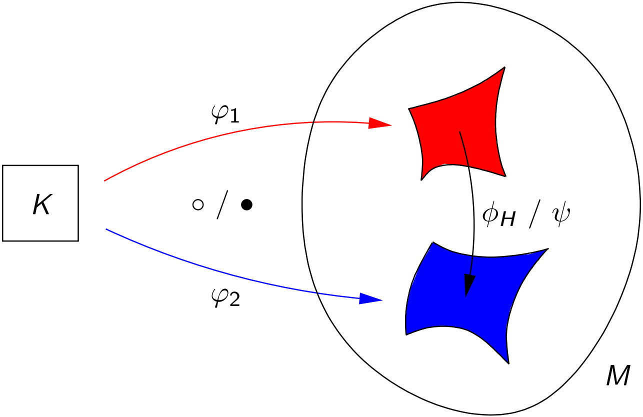
$\varphi_1 \sim_{\text{im}} \varphi_2 \; := \; \varphi_1$ and φ_2 are equivalent $\; :=$

$\exists$ a compactly supported symplectomorphism ψ of (M, ω) s.t.

$$\psi(\varphi_1(K)) \; = \; \varphi_2(K)$$

cf. Gutt–Usher

Of course, $\sim_0 \; \Rightarrow \; \sim_\omega \; \Rightarrow \; \sim_{\text{im}}$


K
φ_1
φ_2
$\circ$ / $\bullet$
ϕ_H / ψ
M

$\sim_0 \Leftrightarrow \sim_\omega$ if M is a starshaped domain in $\mathbb{R}^4$
or $\mathbb{C}\mathrm{P}^2$ or $\mathrm{S}^2 \times \mathrm{S}^2$ (up to swap)

but if the symplectic mapping class group is large, the difference may be large

$\sim_\omega$ and $\sim_{\mathrm{im}}$ are essentially the same if K is a polydisc

$$\mathrm{P}(a_1, \dots, a_n) := \mathrm{D}^2(a_1) \times \cdots \times \mathrm{D}^2(a_n)$$

Lemma E (**Eliashberg**)

Assume that K is a polydisc and that $\varphi_1 \sim_{\mathrm{im}} \varphi_2$.
Then there exists a permutation σ of the coordinates $z_1, \dots, z_n$ such that $\varphi_1 \circ \sigma \sim_\omega \varphi_2$.

Previous results on Question 2

(1) If K is a 4-ellipsoid and M^4 is a ball or a cube, then $\mathrm{Emb}(K, M)$ is connected (**McDuff**)

More generally (**Cristofaro–Gardiner**): $\mathrm{Emb}(X^4_{\Omega_{\mathrm{conc}}}, X^4_{\Omega_{\mathrm{conv}}})$ is connected

Tools:

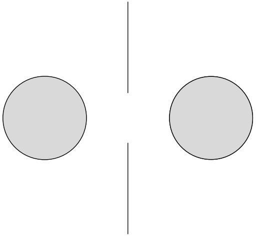
- relation $B^{2n} \overset{s}{\hookrightarrow} (M, \omega)$ $\longleftrightarrow$ symplectic cone of the blow-up of (M, ω)
- symplectic inflation

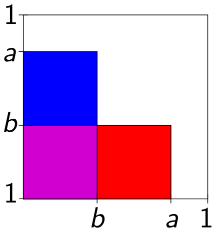
But: Nothing is known about this in dimension ≥ 6 !

(2) **Gromov**'s **camel theorem**: For the camel-space $\mathcal{C}(1)$ in $\mathbb{R}^{2n}$,

$$\mathrm{Emb}(B^{2n}(a), \mathcal{C}(1))$$

is **not** connected for $a > 1$





(3) For $a, b < 1$ with $a + b > 1$ the two embeddings

$$\mathtt{id},\ \sigma(z_1, z_2) = (z_2, z_1)\colon\ \mathrm{P}(a, b) \to \mathring{\mathrm{C}}^4(1)$$

are not isotopic (**Floer–Hofer–Wysocki**)

Other pairs of non-equivalent embeddings were found by **Hind, Gutt–Usher, Dimitroglou–Rizell**

Results

First: in dimension 4

K is always a cube $\mathrm{C}^4(a) = \mathrm{D}^2(a) \times \mathrm{D}^2(a)$

Theorem 1 $M = \mathring{\mathrm{B}}^4(3)$ or $\mathbb{C}\mathrm{P}^2(3)$

(i) Consider the sequence

$$s_n = \frac{1}{g_n^2 + g_{n+1}^2}, \qquad n \geq 0$$

where g_n is the n'th odd-index Fibonacci number. Hence

$$(s_0, s_1, s_2, s_3, \dots) = \left(\frac{1}{2}, \frac{1}{5}, \frac{1}{29}, \frac{1}{194}, \dots \right).$$

Then for $a \in (1, 1+s_n)$ there are at least $n+1$ non-equivalent symplectic embeddings of $\mathrm{C}^4(a)$ into $\mathring{\mathrm{B}}^4$ and $\mathbb{C}\mathrm{P}^2$.

(ii) There are infinitely many non-equivalent symplectic embeddings of $\mathrm{C}^4(1)$ into $\mathring{\mathrm{B}}^4$ and $\mathbb{C}\mathrm{P}^2$.

Proof

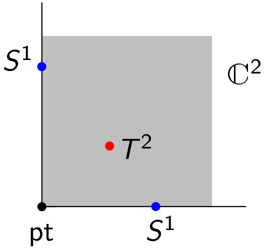
Main idea: use exotic Lagrangian tori and almost toric fibrations

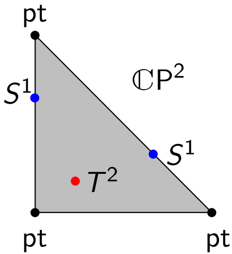
independently by **Chekanov** and **Mikhalkin**

the idea to use the Chekanov torus in $\mathring{B}^4$ to construct an exotic cube embedding also appears in **Gutt–Usher**

Recall: toric fibration: fibers are tori T^2 or subtori

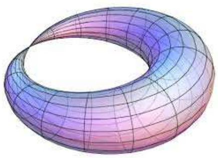
Examples



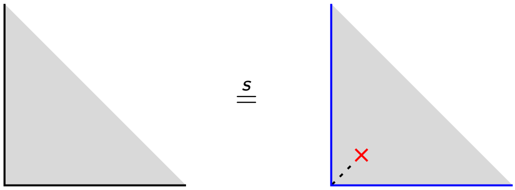


almost toric fibration:

allow also the next best singularity (focus-focus):

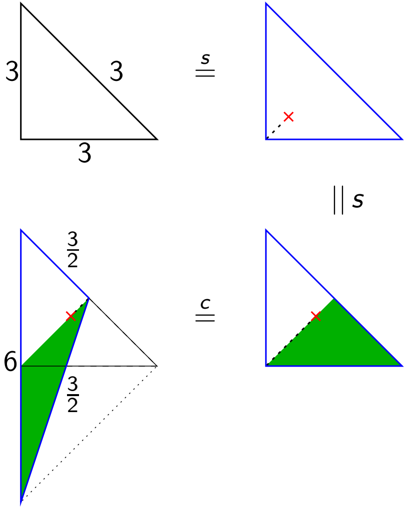


Nguyen Tien Zung and **Margaret Symington**:

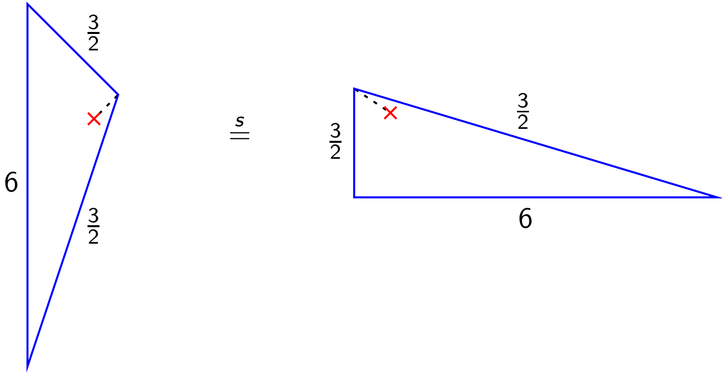


now take $M = \mathbb{C}P^2$

Renato Vianna:



Hence obtain ATF of $\mathbb{C}P^2(3)$ over rectangular triangle:

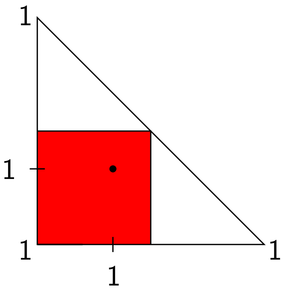


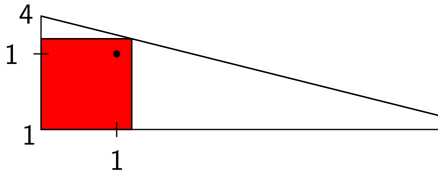
Have a symplectomorphism $\Phi\colon \text{ATF} \to \mathbb{C}P^2$ of $\mathbb{C}P^2$

For $a < 1 + \frac{1}{5}$: $C^4(a) \subset \text{ATF}$

Hence obtain $\varphi := \Phi|_{C^4(a)}\colon C^4(a) \overset{s}{\hookrightarrow} \mathbb{C}P^2$

Also have $\texttt{id}\colon C^4(a) \subset \text{TF} = \mathbb{C}P^2$





Claim: $\mathtt{id} \not\sim_{\mathrm{im}} \varphi$ if $a \in [1, 1+\frac{1}{5})$

Proof: If not, $\exists$ a symplectomorphism ψ of $\mathbb{C}P^2$ such that

$$(\psi \circ \mathtt{id})(C^4(a)) \;=\; \varphi(C^4(a)).$$

By Lemma E, $\exists$ a coordinate permutation σ of $\mathbb{C}^2$ and a symplectomorphism Ψ of $\mathbb{C}P^2$ such that

$$\Psi \circ \mathtt{id} \circ \sigma \;=\; \varphi\colon\; C^4(a) \to \mathbb{C}P^2\,.$$

Restricting to the central torus $L := T(1) \times T(1)$ we obtain

$$\Psi(L) = \varphi(L).$$

But the Clifford torus L and $\varphi(L)$ are not symplectomorphic (**Vianna** and independently **Galkin–Mikhalkin**, or even easier by versal deformations...) $\square$

From $\triangle(\frac{3}{2}, \frac{3}{2}, 6)$ go on by always piercing the vertex opposite to the second longest edge

Get the sequence of Fibonacci triangles $\triangle_n$ containing $\square(a)$ if

$$a \le 1 + \frac{1}{g_n^2 + g_{n+1}^2}$$

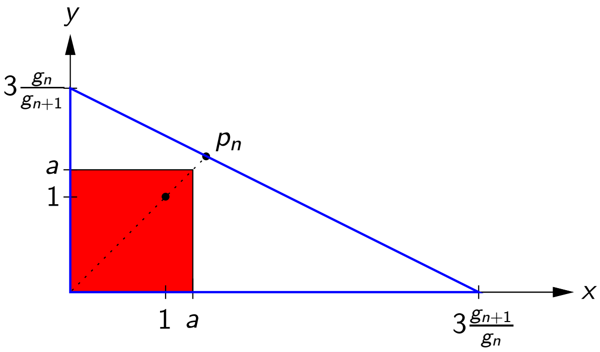


Figure: $\square(a) \subset \triangle_n$

Proof for $M = \mathring{B}^4$

By neck-stretching (**Hind**), or **Casals–Vianna**, can assume that $\varphi(C^4(a))$ is disjoint from the line at infinity, i.e.

$$\varphi\colon\ C^4(a) \to \mathring{B}^4$$

The claim now follows from (i) since $\sim_{\text{im}}$ is defined in terms of compactly supported symplectomorphisms

Theorem 2 $M = C^4(2)$ or $S^2(2) \times S^2(2)$

(i) There exists a sequence t_n decreasing to 0 that starts with $(t_0, t_1, t_2, t_3, \dots) =$

$$\left(1,\ \tfrac{1}{3},\ \tfrac{1}{11},\ \tfrac{1}{29},\ \tfrac{1}{55},\ \tfrac{1}{59},\ \tfrac{1}{89},\ \tfrac{1}{131},\ \tfrac{1}{169},\ \tfrac{1}{181},\ \tfrac{1}{239},\ \tfrac{1}{305},\ \tfrac{1}{335},\ \tfrac{1}{339},\ \tfrac{1}{379},\ \dots\right)$$

and contains the subsequence

$$\frac{1}{4n^2+6n+1}, \qquad n \geq 2, \qquad (*)$$

such that for $a \in (1, 1+t_n)$ there exist at least $n+1$ non-equivalent symplectic embeddings of $C^4(a)$ into $\mathring{C}^4$ and $S^2 \times S^2$.

(ii) There are infinitely many non-equivalent symplectic embeddings of $C^4(1)$ into $\mathring{C}^4$ and $S^2 \times S^2$.

Remark

While the sequence

$$\frac{1}{g_n^2 + g_{n+1}^2}$$

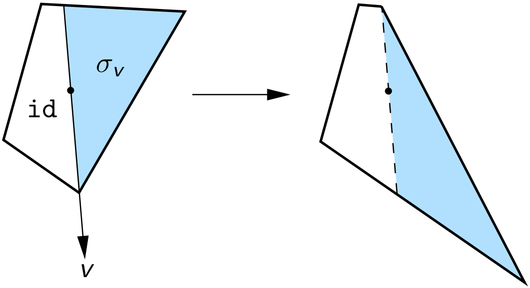
in Theorem 1 decreases like $\frac{1}{2^n}$, the sequence $(*)$ decreases like $\frac{1}{n^2}$

$\rightsquigarrow$ many more inequivalent cubes $\mathrm{C}^4(a)$ in $\mathring{\mathrm{C}}^4$ than in $\mathring{\mathrm{B}}^4$ for given a

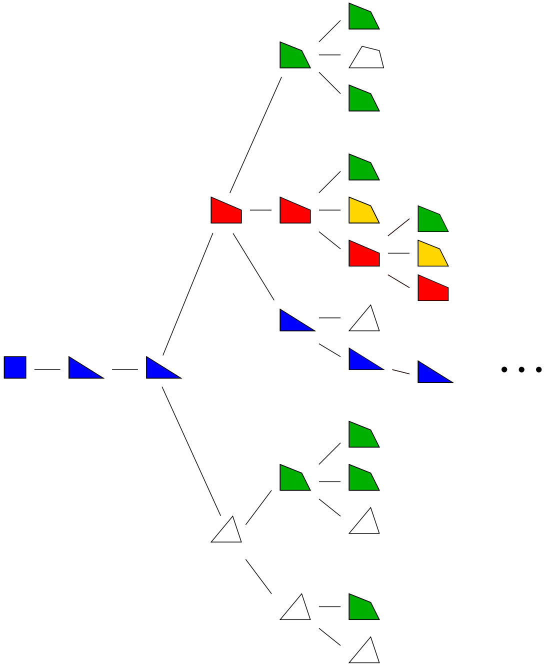
Proof

As before, the case $M = \mathring{\mathbb{C}}^4$ follows from the case $M = S^2 \times S^2$

Look at ATF of $S^2 \times S^2$,
obtained by mutations starting from the TF over the square



We get the following graph (tree ???) of bases of ATF:



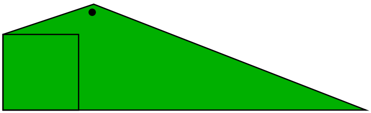
We are happy if we see a basis with a smooth corner such that we can put a square of size > 1 into that corner

All triangles with a smooth corner have this property: Pell triangles but their square-sizes decrease exponentially

Among quadrilaterals with a smooth vertex, there are

good ones: accommodate $\square(1)$

bad ones: do not accommodate $\square(1)$:



The red quadrilaterals are **fat**: give rise to the sequence $\frac{1}{4n^2+6n+1}$

Each red quadrilateral has one **fat** son, one good son, and one bad son

The central tori of the red and yellow quadrilaterals are mutually not symplectomorphic – since the maximal integral weight is increasing – and one can describe their maximal squares $\square(a)$ (**Buc-d'Alché**, see also **Pascaleff–Tonkonog**)

Conjecture The trivalent tree of good quadrilaterals generated by the red and yellow quadrilaterals describes all the non-equivalent cube embeddings into $S^2 \times S^2$

Higher dimensions

Let $(\widehat{M}, \omega)$ be a monotone product of closed toric symplectic manifolds, at least one of whose factors is $\mathbb{C}P^2(3)$ or $S^2(2) \times S^2(2)$. Let (M^{2n}, ω) be a partial affine part of such a manifold.

E.g. $(0,2)^{2n}$, $\mathring{B}^4(3) \times \mathring{D}^2(2)$, $(0,2)^4 \times \mathbb{C}P^2(3) \times S^2(2)$

Theorem 3

(i) There exists a decreasing sequence $c_k \to 1$ such that for $a \in [1, c_k)$ there are at least $k+1$ non-equivalent embeddings

$$C^{2n}(a) \overset{s}{\hookrightarrow} (M, \omega).$$

(ii) There are infinitely many non-equivalent embeddings

$$C^{2n}(1) \overset{s}{\hookrightarrow} (M, \omega).$$

Proof

In the factors $\mathbb{C}P^2$ or $S^2 \times S^2$ use the exotic tori above, and in the other factors use the Clifford torus

Build the product of these tori.

Most of them are not symplectomorphic. $\square$

If there are at least two factors of $\mathbb{C}P^2$ or $S^2 \times S^2$ in $(\widehat{M}, \omega)$, we get many more inequivalent cubes than in dimension 4.

Example For $M = \mathring{C}^{2n}(2)$ the number of inequivalent cubes $C^{2n}(1+s)$ is at least of the order of $\left(\frac{1}{\sqrt{s}}\right)^n$ as $s \to 0$

(**Question** Do there exist non-equivalent embeddings $C^6(a) \overset{s}{\hookrightarrow} \mathring{B}^6$?)

Why are the tori $\varphi_j(T_{\text{Cliff}})$ mutually non-equivalent?

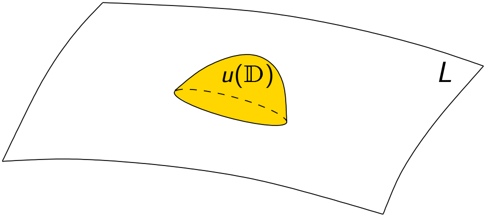
One method:

(**Eliashberg–Polterovich**;
Vianna, **Mikhalkin**, **Pascaleff–Tonkonog**)

Count the number of holomorphic discs

$$u\colon (\mathbb{D}, \partial\mathbb{D}) \to (M, L)$$

of Maslov index 2



Easier method: Versal deformations (**Chekanov**)

Idea: Study a symplectic invariant for tori **nearby** L

Example: **Displacement energy**

For $H\colon [0,1] \times M \to \mathbb{R}$ define

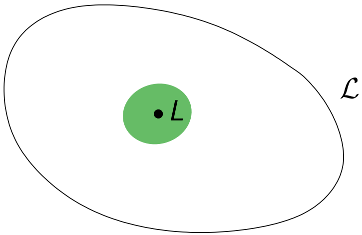
$$\|H\| = \int_0^1 \left(\max_{x \in M} H(t,x) - \min_{x \in M} H(t,x) \right) dt$$

For $A \subset M$ define

$$e(A) = \inf_{H \in \mathcal{H}} \left\{ \|H\| \mid \varphi_H^1(A) \cap A = \varnothing \right\}$$

Exercise: $e(D^2(a)) = e(S^1(a)) = a$

$e(\mathrm{T}_{\mathrm{Cliff}}, \mathbb{C}\mathrm{P}^2) = \infty$, but not at nearby tori:



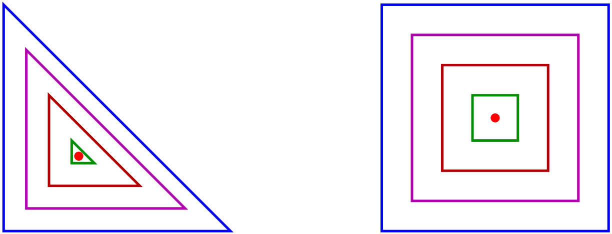
Weinstein: locally

$$\{ \text{ Lagrangian tori near } L \ \} \,/\, \mathrm{Ham} \;=\; H^1(L;\mathbb{R})$$

Since $e \colon \mathcal{L} \to [0,\infty]$ is Ham-invariant, obtain
function germ $e_L \colon (H^1(L;\mathbb{R}),0) \to [0,\infty]$

Harder exercise:

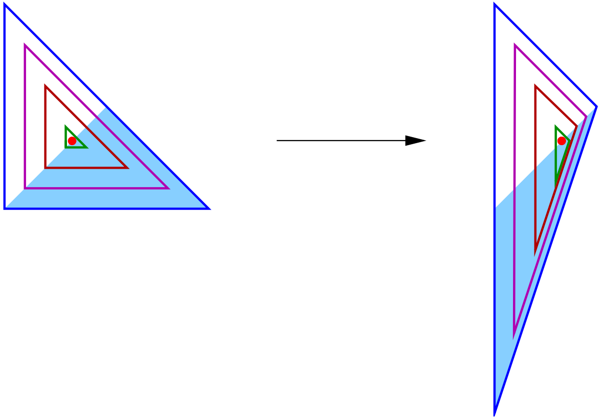
The level lines of $e_{T_{\mathrm{Cliff}}}$ in $\mathbb{C}P^2$ and $S^2 \times S^2$ are:



Hint: Use

$$e\big(S^1(a_1) \times \cdots \times S^1(a_n), \mathbb{R}^{2n}\big) \;=\; \min a_i$$

Since the mutation is done by a half-shear (in $SL(2, \mathbb{Z})$):



(away from thin neighbourhoods of rays!)
i.e. the level lines know the ATF up to $SL(2, \mathbb{Z})$

Hence the set of integral angles of the ATF is an invariant of the central torus