Symplectically knotted cubes

joint with

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The problem

Take a “good” compact set $K \subset (\mathbb{R}^{2n}, \omega_0)$, $\omega_0 = \sum_{j=1}^{n} dx_j \wedge dy_j$
like a ball or a cube of an ellipsoid or a polydisc

$(M^{2n}, \omega)$ connected symplectic manifold

$\text{Emb}_\omega(K, M) := \text{space of symplectic embeddings } K \to M$

Question 1 Is $\text{Emb}_\omega(K, M)$ non-empty?

Question 2 Is $\text{Emb}_\omega(K, M)$ connected?

If not, what is $\pi_0(\text{Emb}_\omega(K, M))$?

Question 3 What is the topology $(\pi_k, H_k)$ of $\text{Emb}_\omega(K, M)$?
Most known results are on Question 1.

Examples

- Gromov’s nonsqueezing theorem:
  \[ B^{2n}(a) \hookrightarrow B^2(A) \times \mathbb{R}^{2n-2} \text{ only if } a \leq A \]

- The ball packing problem \( \bigcup_{k} B^4(1) \hookrightarrow B^4(A) \)
  (Gromov, McDuff–Polterovich, Biran)

- The problem \( E(1, a) \hookrightarrow B^4(A) \) (McDuff–S)

- Many new recent results, but also many open problems
Question 2: Much less is known ...

Question 3: Almost nothing is known

See however Anjos–Lalonde–Pinsonnault
and the recent Chaidez–Munteanu

Precisions on Question 2:

\( K \preceq (M, \omega) \) means:
There exists a symplectic embedding of a neighbourhood of \( K \)

Several equivalence relations on \( \text{Emb}_\omega(K, M) \) make sense:
\( \varphi_1 \sim_0 \varphi_2 \coloneqq \exists \text{ a smooth path } \varphi_t : K \xrightarrow{s} M \text{ connecting } \varphi_1 \text{ with } \varphi_2 \)

For \( K \) simply connected (as always here) this is the same as:

\( \exists \text{ a Hamiltonian isotopy } \phi_H \text{ of } M \text{ such that} \)

\[ \phi_H \circ \varphi_1 = \varphi_2 \]

\( \varphi_1 \sim_\omega \varphi_2 \coloneqq \exists \text{ a compactly supported symplectomorphism } \psi \text{ of } (M, \omega) \text{ s.t.} \)

\[ \psi \circ \varphi_1 = \varphi_2 \]

\( \varphi_1 \sim_{\text{im}} \varphi_2 \coloneqq \varphi_1 \text{ and } \varphi_2 \text{ are equivalent} \coloneqq \exists \text{ a compactly supported symplectomorphism } \psi \text{ of } (M, \omega) \text{ s.t.} \)

\[ \psi(\varphi_1(K)) = \varphi_2(K) \]

cf. Gutt–Usher

Of course, \( \sim_0 \Rightarrow \sim_\omega \Rightarrow \sim_{\text{im}} \)
\( \sim_0 \iff \sim_\omega \) if \( M \) is a starshaped domain in \( \mathbb{R}^4 \) or \( \mathbb{C}P^2 \) or \( S^2 \times S^2 \) (up to swap)

but if the symplectic mapping class group is large, the difference may be large

\( \sim_\omega \) and \( \sim_{\text{im}} \) are essentially the same if \( K \) is a polydisc

\[
P(a_1, \ldots, a_n) := D^2(a_1) \times \cdots \times D^2(a_n)
\]

**Lemma E (Eliashberg)**

*Assume that \( K \) is a polydisc and that \( \varphi_1 \sim_{\text{im}} \varphi_2 \). Then there exists a permutation \( \sigma \) of the coordinates \( z_1, \ldots, z_n \) such that \( \varphi_1 \circ \sigma \sim_\omega \varphi_2 \).*
Previous results on Question 2

(1) If $K$ is a 4-ellipsoid and $M^4$ is a ball or a cube, then $\text{Emb}(K, M)$ is connected (McDuff)

More generally (Cristofaro–Gardiner): $\text{Emb}(X^4_{\Omega_{\text{conc}}}, X^4_{\Omega_{\text{conv}}})$ is connected

Tools:
- relation $B^{2n} \hookrightarrow (M, \omega) \leftrightarrow$ symplectic cone of the blow-up of $(M, \omega)$
- symplectic inflation

But: Nothing is known about this in dimension $\geq 6$!

(2) Gromov’s camel theorem: For the camel-space $C(1)$ in $\mathbb{R}^{2n}$,

$$\text{Emb}(B^{2n}(a), C(1))$$

is not connected for $a > 1$
(3) For $a, b < 1$ with $a + b > 1$ the two embeddings

$$\text{id}, \sigma(z_1, z_2) = (z_2, z_1): P(a, b) \to \hat{C}^4(1)$$

are not isotopic (Floer–Hofer–Wysocki)

Other pairs of non-equivalent embeddings were found by Hind, Gutt–Usher, Dimitroglou–Rizell
Results

First: in dimension 4

$K$ is always a cube $C^4(a) = D^2(a) \times D^2(a)$

Theorem 1  $M = \mathcal{B}^4(3)$ or $\mathbb{C}P^2(3)$

(i) Consider the sequence

$$s_n = \frac{1}{g_n^2 + g_{n+1}^2}, \quad n \geq 0$$

where $g_n$ is the $n$'th odd-index Fibonacci number. Hence

$$(s_0, s_1, s_2, s_3, \ldots) = \left(\frac{1}{2}, \frac{1}{5}, \frac{1}{29}, \frac{1}{194}, \ldots\right).$$

Then for $a \in (1, 1 + s_n)$ there are at least $n + 1$ non-equivalent symplectic embeddings of $C^4(a)$ into $\mathcal{B}^4$ and $\mathbb{C}P^2$.

(ii) There are infinitely many non-equivalent symplectic embeddings of $C^4(1)$ into $\mathcal{B}^4$ and $\mathbb{C}P^2$. 
Proof

Main idea: use exotic Lagrangian tori and almost toric fibrations independently by Chekanov and Mikhalkin
the idea to use the Chekanov torus in $\mathbb{R}^4$ to construct an exotic cube embedding also appears in Gutt–Usher

Recall: toric fibration: fibers are tori $T^2$ or subtori

Examples
almost toric fibration:
allow also the next best singularity (focus-focus):

Nguyen Tien Zung and Margaret Symington:
now take $M = \mathbb{CP}^2$

**Renato Vianna:**

\[ \frac{3}{2} \]

\[ \frac{3}{2} \]

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\[ \frac{3}{2} \]
Hence obtain ATF of $\mathbb{C}P^2(3)$ over rectangular triangle:
Have a symplectomorphism $\Phi: \text{ATF} \to \mathbb{C}P^2$ of $\mathbb{C}P^2$

For $a < 1 + \frac{1}{5}$: $C^4(a) \subset \text{ATF}$

Hence obtain $\varphi := \Phi|_{C^4(a)}: C^4(a) \hookrightarrow \mathbb{C}P^2$

Also have $\text{id}: C^4(a) \subset \text{TF} = \mathbb{C}P^2$
Claim: $\text{id} \not\sim \text{im} \varphi$ if $a \in [1, 1 + \frac{1}{5})$

Proof: If not, $\exists$ a symplectomorphism $\psi$ of $\mathbb{CP}^2$ such that

$$(\psi \circ \text{id})(C^4(a)) = \varphi(C^4(a)).$$

By Lemma E, $\exists$ a coordinate permutation $\sigma$ of $\mathbb{C}^2$ and a symplectomorphism $\Psi$ of $\mathbb{CP}^2$ such that

$$\Psi \circ \text{id} \circ \sigma = \varphi: C^4(a) \to \mathbb{CP}^2.$$

Restricting to the central torus $L := T(1) \times T(1)$ we obtain

$$\Psi(L) = \varphi(L).$$

But the Clifford torus $L$ and $\varphi(L)$ are not symplectomorphic (Vianna and independently Galkin–Mikhalkin, or even easier by versal deformations...)}
From $\triangle(\frac{3}{2}, \frac{3}{2}, 6)$ go on by always piercing the vertex opposite to the second longest edge.

Get the sequence of Fibonacci triangles $\triangle_n$ containing $\square(a)$ if

$$a \leq 1 + \frac{1}{g_n^2 + g_{n+1}^2}$$

Figure: $\square(a) \subset \triangle_n$
Proof for $M = \mathring{B}^4$

By neck-stretching (Hind), or Casals–Vianna, can assume that $\varphi(C^4(a))$ is disjoint from the line at infinity, i.e.

$$\varphi: C^4(a) \to \mathring{B}^4$$

The claim now follows from (i) since $\sim_{im}$ is defined in terms of compactly supported symplectomorphisms.
**Theorem 2** \( M = C^4(2) \) or \( S^2(2) \times S^2(2) \)

(i) There exists a sequence \( t_n \) decreasing to 0 that starts with 
\[
( t_0, t_1, t_2, t_3, \ldots ) = \left( 1, \frac{1}{3}, \frac{1}{11}, \frac{1}{29}, \frac{1}{55}, \frac{1}{59}, \frac{1}{89}, \frac{1}{131}, \frac{1}{169}, \frac{1}{181}, \frac{1}{239}, \frac{1}{305}, \frac{1}{335}, \frac{1}{339}, \frac{1}{379}, \ldots \right)
\]
and contains the subsequence
\[
\frac{1}{4n^2 + 6n + 1}, \quad n \geq 2,
\]

such that for \( a \in (1, 1 + t_n) \) there exist at least \( n + 1 \) non-equivalent symplectic embeddings of \( C^4(a) \) into \( \hat{C}^4 \) and \( S^2 \times S^2 \).

(ii) There are infinitely many non-equivalent symplectic embeddings of \( C^4(1) \) into \( \hat{C}^4 \) and \( S^2 \times S^2 \).
Remark

While the sequence

\[
\frac{1}{g_n^2 + g_{n+1}^2}
\]

in Theorem 1 decreases like \( \frac{1}{2^n} \), the sequence \((*)\) decreases like \( \frac{1}{n^2} \)

\( \sim \) many more inequivalent cubes \( C^4(a) \) in \( \hat{C}^4 \) than in \( \hat{B}^4 \) for given \( a \)
Proof
As before, the case $M = \hat{\mathcal{C}}^4$ follows from the case $M = S^2 \times S^2$

Look at ATF of $S^2 \times S^2$, obtained by mutations starting from the TF over the square

![Diagram of bases of ATF](image)

We get the following graph (tree???) of bases of ATF:
We are happy if we see a basis with a smooth corner such that we can put a square of size $> 1$ into that corner

All triangles with a smooth corner have this property: Pell triangles but their square-sizes decrease exponentially

Among quadrilaterals with a smooth vertex, there are

- good ones: accommodate □(1)
- bad ones: do not accommodate □(1):

The red quadrilaterals are fat: give rise to the sequence $\frac{1}{4n^2+6n+1}$

Each red quadrilateral has one fat son, one good son, and one bad son
The central tori of the red and yellow quadrilaterals are mutually not symplectomorphic – since the maximal integral weight is increasing – and one can describe their maximal squares □(a) (Buc-d’Alché, see also Pascaleff–Tonkonog)

**Conjecture** The trivalent tree of good quadrilaterals generated by the red and yellow quadrilaterals describes all the non-equivalent cube embeddings into $S^2 \times S^2$
Higher dimensions

Let \((\hat{M}, \omega)\) be a monotone product of closed toric symplectic manifolds, at least one of whose factors is \(\mathbb{CP}^2(3)\) or \(S^2(2) \times S^2(2)\). Let \((M^{2n}, \omega)\) be a partial affine part of such a manifold.

E.g. \((0, 2)^{2n}, B^4(3) \times D^2(2), (0, 2)^4 \times \mathbb{CP}^2(3) \times S^2(2)\)

**Theorem 3**

(i) There exists a decreasing sequence \(c_k \to 1\) such that for \(a \in [1, c_k]\) there are at least \(k + 1\) non-equivalent embeddings

\[C^{2n}(a) \leftrightarrow (M, \omega).\]

(ii) There are infinitely many non-equivalent embeddings

\[C^{2n}(1) \leftrightarrow (M, \omega).\]
Proof
In the factors $\mathbb{CP}^2$ or $S^2 \times S^2$ use the exotic tori above, and in the other factors use the Clifford torus
Build the product of these tori.
Most of them are not symplectomorphic.

If there are at least two factors of $\mathbb{CP}^2$ or $S^2 \times S^2$ in $(\hat{M}, \omega)$, we get many more inequivalent cubes than in dimension 4.

Example For $M = \hat{\mathbb{C}}^{2n}(2)$ the number of inequivalent cubes $C^{2n}(1 + s)$ is at least of the order of $\left( \frac{1}{\sqrt{s}} \right)^n$ as $s \to 0$

(Question Do there exist non-equivalent embeddings $C^6(a) \hookrightarrow \hat{\mathbb{B}}^6$ ?)
Why are the tori $\varphi_j(T_{\text{Cliff}})$ mutually non-equivalent?

**One method:**

(Eliashberg–Polterovich; Vianna, Mikhalkin, Pascaleff–Tonkonog)

Count the number of holomorphic discs

$$u: (\mathbb{D}, \partial \mathbb{D}) \to (M, L)$$

of Maslov index 2
Easier method: Versal deformations (Chekanov)

Idea: Study a symplectic invariant for tori nearby $L$

Example: Displacement energy

For $H : [0, 1] \times M \to \mathbb{R}$ define

$$\|H\| = \int_0^1 \left( \max_{x \in M} H(t, x) - \min_{x \in M} H(t, x) \right) dt$$

For $A \subset M$ define

$$e(A) = \inf_{H \in \mathcal{H}} \left\{ \|H\| \mid \varphi_H^1(A) \cap A = \emptyset \right\}$$

Exercise: $e(D^2(a)) = e(S^1(a)) = a$
$e(T_{\text{Cliff}}, \mathbb{C}P^2) = \infty$, but not at nearby tori:

Weinstein: locally

\[
\{ \text{Lagrangian tori near } L \} / \text{Ham} = H^1(L; \mathbb{R})
\]

Since $e : \mathcal{L} \rightarrow [0, \infty]$ is Ham-invariant, obtain function germ $e_L : (H^1(L; \mathbb{R}), 0) \rightarrow [0, \infty]$
**Harder exercise:**
The level lines of $e_{T_{\text{Cliff}}}$ in $\mathbb{C}P^2$ and $S^2 \times S^2$ are:

**Hint:** Use

$$e(S^1(a_1) \times \cdots \times S^1(a_n), \mathbb{R}^{2n}) = \min a_i$$
Since the \textit{mutation} is done by a \textit{half-shear} (in $\text{SL}(2, \mathbb{Z})$):

(away from thin neighbourhoods of rays!)

i.e. \textit{the level lines know the ATF up to $\text{SL}(2, \mathbb{Z})$}

Hence the \textit{set of integral angles} of the ATF is an invariant of the central torus