Actions filtrations associated to smooth categorical compactifications

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Smooth categorical compactifications

Smooth categorical compactifications I

Let $\mathcal{C}$ be a smooth A_∞ category.

Definition

A map of A_∞ categories

$$F : \mathcal{B} \to \mathcal{C} \tag{1}$$

is called a *smooth categorical compactification* if the following conditions hold:

(i) $\mathcal{B}$ is smooth and proper;

(ii) $\ker F$ is split generated by a *finite* collection of objects;

(iii) $\mathcal{B}/\ker F \to \mathcal{C}$ is a Morita equivalence (i.e. an equivalence on $\mathrm{Perf}(-)$)

This notion appears e.g. in recent work of Efimov.

Smooth categorical compactifications II

Smooth categorical compactifications of $\mathcal{C}$ form a category $\mathcal{K}_{/\mathcal{C}}$. A morphism $(\mathcal{B}_1 \to \mathcal{C}) \to (\mathcal{B}_2 \to \mathcal{C})$ is a functor $\phi : \mathcal{B}_1 \to \mathcal{B}_2$ such that

- ϕ is a smooth categorical compactification;
- the following diagram commutes:

$$\begin{array}{ccc} \mathcal{B}_1 & \xrightarrow{\phi} & \mathcal{B}_2 \\ & \searrow & \downarrow \\ & & \mathcal{C} \end{array} \tag{2}$$

We say that $(\mathcal{B}_1 \to \mathcal{C})$ and $(\mathcal{B}_2 \to \mathcal{C})$ are *equivalent up to zig-zag* if there are morphisms

$$(\mathcal{B}_1 \to \mathcal{C}) \leftarrow (\mathcal{B}_{i_1} \to \mathcal{C}) \to (\mathcal{B}_{i_2} \to \mathcal{C}) \leftarrow \dots (\mathcal{B}_{i_n} \to \mathcal{C}) \to (\mathcal{B}_2 \to \mathcal{C}) \tag{3}$$

An example from algebraic geometry

Let U be a smooth open algebraic variety over $\mathbb{C}$. By work of Nagata and Hironaka, there exists an inclusion $i : U \hookrightarrow X$, where X is smooth and proper and $U = X - D$ for D a divisor.

Apply $\mathbf{D}^b_{\mathrm{coh}}(-)$. We get a functor of A_∞ categories

$$i^* : \mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(U). \tag{4}$$

Fact (Thomason–Trobaugh)

The functor (4) is a smooth categorical compactification.

Using the Weak factorization theorem for birational maps (Abramovich, Karu, Matsuki and Wlodarczyk), it can be shown that the smooth categorical compactification (4) is independent of the choice of compactification $i : U \hookrightarrow X$ up to zig-zag.

An example from symplectic topology I

Let X be a Liouville manifold with ideal boundary $(\partial_\infty X, \xi_\infty)$. The *wrapped Fukaya category* $\mathcal{W}(X)$ is an A_∞ category and is an important invariant of X. Given a closed subset $F \subset \partial_\infty X$ (a "stop"), one can also consider the (partially) wrapped Fukaya category of the pair $\mathcal{W}(X, F)$. There is a natural functor

$$\mathcal{W}(X, F) \to \mathcal{W}(X) \tag{5}$$

Fact

Suppose that F is a page of an open book decomposition $\pi : \partial_\infty X \to S^1$. If X is Weinstein, the functor (5) is a smooth categorical compactification.

Both the formulation and the proof of this fact rely on deep properties of wrapped Fukaya categories of Weinstein manifolds which are mainly due to Ganatra–Pardon–Shende (building on contributions from many other authors).

An example from symplectic topology II

According to celebrated work of Giroux, any contact manifold admits an open book decomposition with Weinstein pages. Hence $\mathcal{W}(X)$ always admits a smooth categorical compactification.

What about uniqueness? Let F_1 be another page arising from a possibly different open book decomposition. There is a quasi-equivalence $\mathcal{W}(X, F_i) \to \mathcal{W}(X, \mathfrak{c}_i)$, where $\mathfrak{c}_i = \text{core}\, F_i$ is isotropic. Up to perturbing the $\mathfrak{c}_i$ (which does not affect the Fukaya category), we have a zig-zag diagram

$$\begin{array}{ccccc} & & \mathcal{W}(X, \mathfrak{c}_0 \cup \mathfrak{c}_1) & & \\ & \swarrow & & \searrow & \\ \mathcal{W}(X, \mathfrak{c}_0) & & & & \mathcal{W}(X, \mathfrak{c}_1) \\ & \searrow & & \swarrow & \\ & & \mathcal{W}(X) & & \end{array} \tag{6}$$

An example from symplectic topology III

Annoying technical point: actually we do not know how to prove that $\mathcal{W}(X, \mathfrak{c}_0 \cup \mathfrak{c}_1)$ is proper (although we strongly expect that it is). However, this does not matter in practice as long as $\mathcal{W}(X, \mathfrak{c}_i)$ is proper for $i = 0, 1$.

On a different note, there is no need to restrict ourselves in the above discussion to stops arising from open book decompositions.

Definition

A stop $\mathfrak{c}$ is said to be *compactifying* if it deforms to an isotropic (i.e. "mostly Legendrian") stop and $\mathcal{W}(X, \mathfrak{c})$ is proper (and a fortiori smooth).

We saw that pages of Weinsteian open books give compactifying stops. Here is a different example: suppose $X = T^*M$ and let $\mathfrak{c}$ be the union of the conormals of a Whitney triangulation. Then $\mathfrak{c}$ is compactifying.

Filtered A_∞ categories and growth functions

Scaling equivalence

Definition

Given a pair of functions $f_1, f_2 : \mathbb{N} \to \mathbb{R}$, we say that f_1, f_2 are *scaling equivalent* and write $f_1 \sim_s f_2$ if there exist constants $C_1, C_2 > 1$ such that $f_1(C_1 n) + C_2 > f_2(n)$ and $f_2(C_1 n) + C_2 > f_1(n)$.

Example

Two polynomials are scaling equivalent iff they have the same degree.

Example

The function $n \mapsto e^n$ is not scaling equivalent to any polynomial.

Filtered A_∞ categories

Definition

A *filtered* A_∞ category $\mathcal{B} = (\mathcal{B}, F^p(-))$ is the data of an (increasing, integral) filtration $F^p \hom(K, L)$ for all pairs of objects $K, L \in \mathcal{B}$. The operations μ^k are required to respect the filtration.

For $k = 2$, the last condition means that $\mu^2(b, a) \in F^{p+q} \hom(K, M)$ if $a \in F^p \hom(K, L)$ and $b \in F^q \hom(L, M)$.

If $F^p C$ is an (increasing, integral) filtration, then

$$F^p H^*(C) := \mathrm{im}(H^*(F^p C) \to H^*(C)). \tag{7}$$

If $\mathcal{B}$ is a filtered A_∞, then the cohomology category $H^*(\mathcal{B})$ is naturally filtered according to (7).

The growth function

Definition

Let $\mathcal{B}$ be a filtered A_∞ category. Given a pair of objects, we let

$$\Gamma_{K,L} : \mathbb{N} \to \mathbb{Z} \tag{8}$$

$$p \mapsto \dim F^p H^* \hom_{\mathcal{B}}(K, L) = F^p \hom_{H^*\mathcal{B}}(K, L). \tag{9}$$

We call $\Gamma_{K,L}$ the *growth function* of K, L. It depends on $\mathcal{B}$ as a filtered A_∞ category.

The main construction

Let $F : \mathcal{B} \to \mathcal{C}$ be a smooth compactification. (Assume moreover that $\mathcal{B}, \mathcal{C}$ are pre-triangulated). By definition, we may choose $D_1, \ldots, D_k \in \ker F$ such that the induced map $\mathcal{B}/\{D_1, \ldots, D_k\} \to \mathcal{C}$ is a Morita equivalence.

Fact

$\mathcal{B}/\{D_1, \ldots, D_k\}$ is naturally a filtered A_∞ category (Lyubashenko–Ovsienko model).

Theorem (C–Kartal)

With the notation as above:

- *$\Gamma_{K,L}$ depends only on $\mathcal{B} \to \mathcal{C}$ up to scaling equivalence (i.e. $\Gamma_{K,L}$ is independent of the choice $D_1, \ldots, D_k$)*
- *In fact, $\Gamma_{K,L}$ only on $\mathcal{B} \to \mathcal{C}$ up to zig-zag of smooth compactifications.*

Computations

An interlude on colimits

Let

$$C_1 \to C_2 \to C_3 \to \dots \tag{10}$$

be a sequence of morphisms of vector spaces (a "filtered directed system"). Then colim C_i is naturally filtered: just let

$$F^p \operatorname{colim} C_i := \operatorname{im}(C_p \to \operatorname{colim} C_i). \tag{11}$$

We will refer to this as the ”colimit filtration".

Examples from algebraic geometry

Let U be an open smooth algebraic variety. Let X be a smooth compactification where $U = X - D$. Recall that $\mathbf{D}^b_{\mathrm{coh}}(X) \to \mathbf{D}^b_{\mathrm{coh}}(U)$ is a smooth categorical compactification.

Fact

Given $F, G \in \mathbf{D}^b_{\mathrm{coh}}(U)$, there exist $\overline{F}, \overline{G} \in \mathbf{D}^b_{\mathrm{coh}}(X)$ such that $F = i^*\overline{F}$ and $G = i^*\overline{G}$ where $i : U \hookrightarrow X$ is the inclusion. Moreover,

$$Ext^*_U(F, G) = \mathrm{colim}\, Ext^*_X(\overline{F}, \overline{G} \otimes O_X(nD)). \tag{12}$$

Let $\Gamma^{Ext}_{F,G}$ be the growth function associated to the colimit filtration.

Proposition

We have $\Gamma_{F,G} \sim_s \Gamma^{Ext}_{F,G}$ (i.e. $\Gamma_{F,G}$ agrees up to scaling equivalence with $\Gamma^{Ext}_{F,G}$).

Examples from symplectic topology

We have the same story for wrapped Fukaya categories. Let (X, λ) be a Liouville manifold and let $K, L \subset X$ be Lagrangians (cylindrical at infinity). First of all, note that

$$HW(K, L) = \operatorname{colim}_n HF(\phi_{nH} K, L), \tag{13}$$

where H is any Hamiltonian which is cylindrical, linear at infinity. Let $\Gamma^{Ham}_{K,L}$ be the associated growth function. $\Gamma^{Ham}_{K,L}$ was studied by McLean, Frauenfelder–Labrousse–Schlenk, Alves–Meiwes, and others.

One similarly has

$$SH(X) = \operatorname{colim}_n HF(X, nH). \tag{14}$$

We denote the associated growth function Γ^{Ham}_{SH}. The study of this invariant was pioneered by McLean (building on work of Seidel).

Let us now choose a compactifying stop $F \subset \partial_\infty X$ and consider the smooth categorical compactification $\mathcal{W}(X,F) \to \mathcal{W}(X)$.

Proposition

The growth function $\Gamma_{K,L}$ agrees up to scaling equivalence with $\Gamma_{K,L}^{Ham}$.

In other words, the abstract growth function associated to the categorical compactification $\mathcal{W}(X,F) \to \mathcal{W}(X)$ recovers the growth function which is defined using Hamiltonians.

Example

Let M be a closed pointed manifold. Fix an auxiliary Riemannian metric and let $\Omega_{\leq l} M$ be the space of based loops of length at most l. Then $\{\mathrm{im}(H_*(\Omega_{\leq p} M) \to H_*(\Omega M))\}$ is a filtered directed system and we let Γ_M^{top} be the associated growth function.
Let $L \subset T^*M$ be a cotangent fiber. McLean proved that $\Gamma_{L,L}^{Ham} \sim^s \Gamma_M^{top}$.
Combining this with the previous proposition, we find that $\Gamma_{K,L} \sim^s \Gamma_M^{top}$.

Spherical functors

In fact, the previous two slides are (at least morally) special cases of a more general statement:

Let $G : \mathcal{A} \to \mathcal{B}$ be a *spherical functor*. Let $S : \mathcal{B} \to \mathcal{B}$ be the *spherical twist*. Consider the filtered directed system $\hom(K, L) \to \hom(K, S(L)) \to \hom(K, S^2(L)) \to \dots$ Let $\Gamma^{sph}_{K,L}$ be the associated growth function.

Theorem

Let $Q : \mathcal{B} \to \mathcal{C}$ be a smooth categorical compactification. If the image of G contains a spit-generator of $\ker Q$, then $\Gamma_{K,L}$ is scaling equivalent to $\Gamma^{sph}_{K,L}$.

Remark

In the preceding examples, we have $S = - \otimes O_X(D)$ and $S = \text{wrap-once}(\textit{negatively})$.

Applications

Homological mirror symmetry for pairs I

Let $(X, \mathfrak{c})$ be a Weinstein pair which is mirror to $(U = Y - D)$. I.e.

$$\begin{array}{ccc} \operatorname{Perf} \mathcal{W}(X, \mathfrak{c}) & \longrightarrow & \operatorname{Perf} \mathcal{W}(X) \\ \| & & \| \\ \mathbf{D}^b_{\mathrm{coh}}(Y) & \longrightarrow & \mathbf{D}^b_{\mathrm{coh}}(U) \end{array} \tag{15}$$

Suppose that $K, L \in \operatorname{Perf} \mathcal{W}(X)$ are mirror to $\mathcal{F}_K, \mathcal{F}_L \in \mathbf{D}^b_{\mathrm{coh}}(U)$ and let $\overline{\mathcal{F}}_K, \overline{\mathcal{F}}_L \in \mathbf{D}^b_{\mathrm{coh}}(X)$ be extensions.

Homological mirror symmetry for pairs II

Theorem

We have $\Gamma_{K,L} \sim_s \Gamma^{Ext}_{\mathcal{F},\mathcal{F}_L}$, *where* $\Gamma^{Ext}_{\mathcal{F},\mathcal{F}_L}$ *is the growth function associated to the filtered directed system* $\{\mathrm{Ext}^*(\overline{\mathcal{F}}_K, \overline{\mathcal{F}}_L \otimes \mathcal{O}_X(D)^{\otimes k})\}_{k\in\mathbb{N}}$. *Hence also (by the previously discussed results)* $\Gamma^{Ham}_{K,L} \sim_s \Gamma^{Ext}_{\mathcal{F},\mathcal{F}_L}$.

Theorem

The growth function of symplectic cohomology Γ^{ham}_{SH} *(as considered by McLean, Seidel) is scaling equivalent to the growth of the f.d.s.* $\{H^*(\Omega^k_X[k] \otimes \mathcal{O}_X(2D))\}$.

Remark

In the proofs, we use the fact that $\Gamma_{K,L}$ is independent of the choice of compactifying stop.

An example: HMS for the projective plane and the binodal cubic

Let $Y = \mathbb{CP}^2$ and let D be the union of a line and a conic. Let $U = Y - D$. Mirror symmetry for this example has been studied by many mathematicians and physicists. Homological mirror symmetry for pairs holds in this setting (Pascaleff, Hacking–Keating). The Weinstein mirror is $X = \{(u, v) \mid uv \neq 1\}$.

Corollary

Let $K, L \in \mathcal{W}(X)$ and let $\mathcal{F}_L, \mathcal{F}_K \in \mathbf{D}^b_{\mathrm{coh}}(U)$ be the mirror sheaves. Then

$$\frac{\log \Gamma^{ham}_{K,L}}{\log n} = \frac{\log \Gamma_{K,L}}{\log n} = \dim(\operatorname{supp} \mathcal{F}_K \cap \operatorname{supp} \mathcal{F}_L). \tag{16}$$

Remark

The proof of this corollary uses the fact that D is ample. In particular, it immediately generalizes to higher dimensions if one has HMS for pairs.

Relation to entropy

Let $\mathcal{B}$ be a smooth and proper (idempotent complete, pre-triangulated) category and let $S : \mathcal{B} \to \mathcal{B}$ be an endofunctor. The *entropy* of S is $\limsup_n \frac{1}{n} \log \dim Ext^*(G, F^n G)$ for G any split generator (Dimitrov–Haiden–Kontsevich–Katzarkov). There is also a notion of *slow entropy*: $\limsup_n \frac{1}{\log n} \log \dim Ext^*(G, F^n G)$ (Fan–Fu–Ouchi).

Suppose $\phi : M \to M$ is a diffeomorphism; let

$$\sup_{N \subset M} \limsup_n \frac{\log \mathrm{Vol}_g(\phi(N))}{n} \tag{17}$$

be the *volume growth* (this agrees with the topological entropy by results of Yomdin and Newhouse). Similarly, we can consider the *slow volume growth*

$$\sup_{N \subset M} \limsup_n \frac{\log \mathrm{Vol}_g(\phi(N))}{\log n} \tag{18}$$

The growth of $\Gamma_{K,L}$ gives lower bounds on these various entropies. This uses work of Frauenfelder–Schlenk, Alves–Meiwes, Sylvan, and others.

Questions

Other categorical compactifications I

- Instead of working with $\mathcal{W}(-)$, we can work with categories of microlocal sheaves (this is essentially equivalent by work of Ganatra–Pardon–Shende);
- In some cases, if Y is a singular variety and $f : X \to Y$ is a resolution, then $f_* : \mathbf{D}^b_{\text{coh}}(X) \to \mathbf{D}^b_{\text{coh}}(Y)$ is a smooth categorical compactification. We unfortunately don't have any good computations in this setting.
- We can consider smooth categorical compactifications of wrapped Fukaya categories of Liouville sectors or (equivalently) Liouville pairs.

Example

Suppose that $\ell \subset S^3$ is a knot and let $\Lambda \subset S^*S^3$ be the conormal of ℓ. Then we can embed Λ into the conormal of a triangulation of S^3; this gives a smooth categorical compactification of $\mathcal{W}(T^*S^3, \Lambda)$.
If $L \subset T^*S^3$ is a cotangent fiber (with $\partial_\infty L \cap \Lambda = \emptyset$), what does $\Gamma_{L,L}$ remember about ℓ? It is related to other knot invariants?

Other categorical compactifications II

At the moment, we only know how to construct smooth categorical compactifications for $\mathcal{W}(X, \mathfrak{c})$ if $X = T^*M$.

Question

Suppose that X is a Liouville sector? Does X always admit a compactifying stop? (i.e. a mostly Legendrian (up to deformation) stop $\mathfrak{f}$ such that $\mathcal{W}(X, \mathfrak{f}) \to \mathcal{W}(X)$ is a smooth categorical compactification).

If X is a Liouville manifold, we saw that this follows from Giroux's work.

Integrality I

Let X be Weinstein and let K, L be Lagrangian submanifolds (cylindrical at infinity).

Question

1. *if $\Gamma_{K,L}$ grows faster than a polynomial, does this imply that it grows exponentially?*
2. *if $\Gamma_{K,L}$ grows slower than a polynomial, does this mean that*

$$\lim \log \Gamma_{K,L} / \log n \in \mathbb{Z} \tag{19}$$

   (i.e. the growth rate is integral).

Integrality II

Here is some (extremely slim) evidence:

1. if $X = T^*M$, then Perf $\mathcal{W}(X) =$ Perf $C_{-*}(\Omega M)$. Now there is a (somewhat) related dichotomy in rational homotopy theory which states that the growth of $\sum_{i \neq n} \dim H_i(\Omega M)$ as a function of n is either polynomial or exponential.
2. if X is mirror to the complement of an anticanonical divisor, then this follows from the previous slide.

Remark

You can also ask similar questions about entropy/polynomial entropy (e.g. is the entropy of the "wrap-once functor" on a Weinstein pair either polynomial or exponential; is the volume growth rate of a Reeb flow either polynomial or exponential, etc.)

Thank you very much for your attention!