Actions filtrations associated to smooth categorical compactifications

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Smooth categorical compactifications
Let $C$ be a smooth $A_{\infty}$ category.

**Definition**

A map of $A_{\infty}$ categories

$$F : \mathcal{B} \rightarrow C$$

(1)

is called a *smooth categorical compactification* if the following conditions hold:

(i) $\mathcal{B}$ is smooth and proper;
(ii) $\ker F$ is split generated by a *finite* collection of objects;
(iii) $\mathcal{B}/\ker F \rightarrow C$ is a Morita equivalence (i.e. an equivalence on Perf($-$))

This notion appears e.g. in recent work of Efimov.
Smooth categorical compactifications of \( \mathcal{C} \) form a category \( \mathcal{K}_{/C} \). A morphism \((\mathcal{B}_1 \to \mathcal{C}) \to (\mathcal{B}_2 \to \mathcal{C})\) is a functor \( \phi : \mathcal{B}_1 \to \mathcal{B}_2 \) such that

- \( \phi \) is a smooth categorical compactification;
- the following diagram commutes:

\[
\begin{array}{ccc}
\mathcal{B}_1 & \xrightarrow{\phi} & \mathcal{B}_2 \\
\downarrow & & \downarrow \\
\mathcal{C} & & \mathcal{C}
\end{array}
\]  

(2)

We say that \((\mathcal{B}_1 \to \mathcal{C})\) and \((\mathcal{B}_2 \to \mathcal{C})\) are *equivalent up to zig-zag* if there are morphisms

\[
(\mathcal{B}_1 \to \mathcal{C}) \leftrightarrow (\mathcal{B}_{i_1} \to \mathcal{C}) \to (\mathcal{B}_{i_2} \to \mathcal{C}) \leftrightarrow \ldots (\mathcal{B}_{i_n} \to \mathcal{C}) \to (\mathcal{B}_2 \to \mathcal{C})
\]  

(3)
Let $U$ be a smooth open algebraic variety over $\mathbb{C}$. By work of Nagata and Hironaka, there exists an inclusion $i : U \hookrightarrow X$, where $X$ is smooth and proper and $U = X - D$ for $D$ a divisor.

Apply $\mathcal{D}^b_{\text{coh}}(-)$. We get a functor of $A_\infty$ categories

$$i^* : \mathcal{D}^b_{\text{coh}}(X) \to \mathcal{D}^b_{\text{coh}}(U).$$

(4)

**Fact (Thomason–Trobaugh)**

*The functor (4) is a smooth categorical compactification.*

Using the Weak factorization theorem for birational maps (Abramovich, Karu, Matsuki and Wlodarczyk), it can be shown that the smooth categorical compactification (4) is independent of the choice of compactification $i : U \hookrightarrow X$ up to zig-zag.
Let $X$ be a Liouville manifold with ideal boundary $(\partial_{\infty} X, \xi_{\infty})$. The wrapped Fukaya category $\mathcal{W}(X)$ is an $A_{\infty}$ category and is an important invariant of $X$. Given a closed subset $F \subset \partial_{\infty} X$ (a “stop”), one can also consider the (partially) wrapped Fukaya category of the pair $\mathcal{W}(X, F)$. There is a natural functor

$$\mathcal{W}(X, F) \to \mathcal{W}(X) \quad (5)$$

**Fact**

*Suppose that $F$ is a page of an open book decomposition $\pi : \partial_{\infty} X \to S^1$. If $X$ is Weinstein, the functor (5) is a smooth categorical compactification.*

Both the formulation and the proof of this fact rely on deep properties of wrapped Fukaya categories of Weinstein manifolds which are mainly due to Ganatra–Pardon–Shende (building on contributions from many other authors).
According to celebrated work of Giroux, any contact manifold admits an open book decomposition with Weinstein pages. Hence $\mathcal{W}(X)$ always admits a smooth categorical compactification.

What about uniqueness? Let $F_1$ be another page arising from a possibly different open book decomposition. There is a quasi-equivalence $\mathcal{W}(X, F_i) \to \mathcal{W}(X, c_i)$, where $c_i =$ core $F_i$ is isotropic. Up to perturbing the $c_i$ (which does not affect the Fukaya category), we have a zig-zag diagram

$$
\begin{array}{ccc}
\mathcal{W}(X, c_0 \cup c_1) & \xleftarrow{} & \mathcal{W}(X, c_0) \\
\mathcal{W}(X, c_0) & \xrightarrow{} & \mathcal{W}(X, c_1) \\
\mathcal{W}(X) & \xleftarrow{} & \mathcal{W}(X) \\
\end{array}
$$

(6)
Annoying technical point: actually we do not know how to prove that $\mathcal{W}(X, c_0 \cup c_1)$ is proper (although we strongly expect that it is). However, this does not matter in practice as long as $\mathcal{W}(X, c_i)$ is proper for $i = 0, 1$.

On a different note, there is no need to restrict ourselves in the above discussion to stops arising from open book decompositions.

**Definition**

A stop $c$ is said to be *compactifying* if it deforms to an isotropic (i.e. "mostly Legendrian") stop and $\mathcal{W}(X, c)$ is proper (and a fortiori smooth).

We saw that pages of Weinsteinian open books give compactifying stops. Here is a different example: suppose $X = T^*M$ and let $c$ be the union of the conormals of a Whitney triangulation. Then $c$ is compactifying.
Filtered $A_\infty$ categories and growth functions
Definition

Given a pair of functions $f_1, f_2 : \mathbb{N} \rightarrow \mathbb{R}$, we say that $f_1, f_2$ are scaling equivalent and write $f_1 \sim_s f_2$ if there exist constants $C_1, C_2 > 1$ such that $f_1(C_1n) + C_2 > f_2(n)$ and $f_2(C_1n) + C_2 > f_1(n)$.

Example

Two polynomials are scaling equivalent iff they have the same degree.

Example

The function $n \mapsto e^n$ is not scaling equivalent to any polynomial.
Definition

A filtered $A_\infty$ category $\mathcal{B} = (\mathcal{B}, F^p(-))$ is the data of an (increasing, integral) filtration $F^p \text{hom}(K, L)$ for all pairs of objects $K, L \in \mathcal{B}$. The operations $\mu^k$ are required to respect the filtration.

For $k = 2$, the last condition means that $\mu^2(b, a) \in F^{p+q} \text{hom}(K, M)$ if $a \in F^p \text{hom}(K, L)$ and $b \in F^q \text{hom}(L, M)$.

If $F^p C$ is an (increasing, integral) filtration, then

$$F^p H^*(C) := \text{im}(H^*(F^p C) \to H^*(C)).$$

(7)

If $\mathcal{B}$ is a filtered $A_\infty$, then the cohomology category $H^*(\mathcal{B})$ is naturally filtered according to (7).
The growth function

Definition

Let $B$ be a filtered $A_\infty$ category. Given a pair of objects, we let

$$\Gamma_{K,L} : \mathbb{N} \rightarrow \mathbb{Z}$$

$$p \mapsto \dim F^p H^* \text{hom}_B(K, L) = F^p \text{hom}_{H^*B}(K, L).$$

We call $\Gamma_{K,L}$ the growth function of $K, L$. It depends on $B$ as a filtered $A_\infty$ category.
The main construction

Let $F : \mathcal{B} \to \mathcal{C}$ be a smooth compactification. (Assume moreover that $\mathcal{B}, \mathcal{C}$ are pre-triangulated). By definition, we may choose $D_1, \ldots, D_k \in \ker F$ such that the induced map $\mathcal{B}/\{D_1, \ldots, D_k\} \to \mathcal{C}$ is a Morita equivalence.

**Fact**

$\mathcal{B}/\{D_1, \ldots, D_k\}$ is naturally a filtered $A_\infty$ category (Lyubashenko–Ovsienko model).

**Theorem (C–Kartal)**

With the notation as above:

- $\Gamma_{K, L}$ depends only on $\mathcal{B} \to \mathcal{C}$ up to scaling equivalence (i.e. $\Gamma_{K, L}$ is independent of the choice $D_1, \ldots, D_k$)
- In fact, $\Gamma_{K, L}$ only on $\mathcal{B} \to \mathcal{C}$ up to zig-zag of smooth compactifications.
Compuations
Let

\[ C_1 \rightarrow C_2 \rightarrow C_3 \rightarrow \ldots \]  \hspace{1cm} (10)

be a sequence of morphisms of vector spaces (a "filtered directed system"). Then \( \text{colim} \ C_i \) is naturally filtered: just let

\[ F^p \text{colim} \ C_i := \text{im}(C_p \rightarrow \text{colim} \ C_i). \]  \hspace{1cm} (11)

We will refer to this as the "colimit filtration".
Examples from algebraic geometry

Let $U$ be an open smooth algebraic variety. Let $X$ be a smooth compactification where $U = X - D$. Recall that $D^b_{\text{coh}}(X) \to D^b_{\text{coh}}(U)$ is a smooth categorical compactification.

Fact

Given $F, G \in D^b_{\text{coh}}(U)$, there exist $\overline{F}, \overline{G} \in D^b_{\text{coh}}(X)$ such that $F = i^* \overline{F}$ and $G = i^* \overline{G}$ where $i : U \hookrightarrow X$ is the inclusion. Moreover,

$$\text{Ext}^*_U(F, G) = \text{colim} \text{Ext}^*_X(\overline{F}, \overline{G} \otimes O_X(nD)).$$  \hspace{1cm} (12)

Let $\Gamma^\text{Ext}_{F,G}$ be the growth function associated to the colimit filtration.

Proposition

We have $\Gamma_{F,G} \sim_s \Gamma^\text{Ext}_{F,G}$ (i.e. $\Gamma_{F,G}$ agrees up to scaling equivalence with $\Gamma^\text{Ext}_{F,G}$).
We have the same story for wrapped Fukaya categories. Let \((X, \lambda)\) be a Liouville manifold and let \(K, L \subset X\) be Lagrangians (cylindrical at infinity). First of all, note that

\[
HW(K, L) = \colim_n HF(\phi_{nH} K, L),
\] (13)

where \(H\) is any Hamiltonian which is cylindrical, linear at infinity. Let \(\Gamma_{K,L}^{Ham}\) be the associated growth function. \(\Gamma_{K,L}^{Ham}\) was studied by McLean, Frauenfelder–Labrousse–Schlenk, Alves–Meiwes, and others.

One similarly has

\[
SH(X) = \colim_n HF(X, nH).
\] (14)

We denote the associated growth function \(\Gamma_{SH}^{Ham}\). The study of this invariant was pioneered by McLean (building on work of Seidel).
Let us now choose a compactifying stop \( F \subset \partial_{\infty} X \) and consider the smooth categorical compactification \( \mathcal{W}(X, F) \to \mathcal{W}(X) \).

**Proposition**

The growth function \( \Gamma_{K,L} \) agrees up to scaling equivalence with \( \Gamma_{Ham}^{K,L} \).

In other words, the abstract growth function associated to the categorical compactification \( \mathcal{W}(X, F) \to \mathcal{W}(X) \) recovers the growth function which is defined using Hamiltonians.

**Example**

Let \( M \) be a closed pointed manifold. Fix an auxiliary Riemannian metric and let \( \Omega_{\leq l} M \) be the space of based loops of length at most \( l \). Then \( \{ \text{im}(H_*(\Omega_{\leq l} M) \to H_*(\Omega M)) \} \) is a filtered directed system and we let \( \Gamma_{top}^M \) be the associated growth function.

Let \( L \subset T^*M \) be a cotangent fiber. McLean proved that \( \Gamma_{Ham}^{L,L} \sim^s \Gamma_{top}^M \).

Combining this with the previous proposition, we find that \( \Gamma_{K,L} \sim^s \Gamma_{top}^M \).
Spherical functors

In fact, the previous two slides are (at least morally) special cases of a more general statement:

Let $G : \mathcal{A} \to \mathcal{B}$ be a spherical functor. Let $S : \mathcal{B} \to \mathcal{B}$ be the spherical twist. Consider the filtered directed system
$$\text{hom}(K, L) \to \text{hom}(K, S(L)) \to \text{hom}(K, S^2(L)) \to \ldots$$
Let $\Gamma_{K,L}^{\text{sph}}$ be the associated growth function.

Theorem

Let $Q : \mathcal{B} \to \mathcal{C}$ be a smooth categorical compactification. If the image of $G$ contains a spit-generator of $\ker Q$, then $\Gamma_{K,L}$ is scaling equivalent to $\Gamma_{K,L}^{\text{sph}}$.

Remark

In the preceding examples, we have $S = - \otimes O_X(D)$ and $S = \text{wrap-once}(\text{negatively})$. 
Applications
Let \((X, c)\) be a Weinstein pair which is mirror to \((U = Y - D)\). I.e.

\[
\begin{align*}
\text{Perf } \mathcal{W}(X, c) & \rightarrow \text{Perf } \mathcal{W}(X) \\
\downarrow & \quad \downarrow \\
\mathbf{D}^b_{\text{coh}}(Y) & \rightarrow \mathbf{D}^b_{\text{coh}}(U)
\end{align*}
\]

(15)

Suppose that \(K, L \in \text{Perf } \mathcal{W}(X)\) are mirror to \(\mathcal{F}_K, \mathcal{F}_L \in \mathbf{D}^b_{\text{coh}}(U)\) and let \(\overline{\mathcal{F}}_K, \overline{\mathcal{F}}_L \in \mathbf{D}^b_{\text{coh}}(X)\) be extensions.
Theorem

We have $\Gamma_{K,L} \sim_s \Gamma_{\mathcal{F},\mathcal{F}_L}^{\text{Ext}}$, where $\Gamma_{\mathcal{F},\mathcal{F}_L}^{\text{Ext}}$ is the growth function associated to the filtered directed system $\{\text{Ext}^*(\mathcal{F}_K, \mathcal{F}_L \otimes O_X(D)^{\otimes k})\}_{k \in \mathbb{N}}$. Hence also (by the previously discussed results) $\Gamma_{\text{Ham}}_{K,L} \sim_s \Gamma_{\mathcal{F},\mathcal{F}_L}^{\text{Ext}}$.

Theorem

The growth function of symplectic cohomology $\Gamma_{\text{SH}}^{\text{ham}}$ (as considered by McLean, Seidel) is scaling equivalent to the growth of the f.d.s. $\{H^*(\Omega_X^k[k] \otimes O_X(2D))\}$.

Remark

In the proofs, we use the fact that $\Gamma_{K,L}$ is independent of the choice of compactifying stop.
An example: HMS for the projective plane and the binodal cubic

Let $Y = \mathbb{CP}^2$ and let $D$ be the union of a line and a conic. Let $U = Y - D$. Mirror symmetry for this example has been studied by many mathematicians and physicists. Homological mirror symmetry for pairs holds in this setting (Pascaleff, Hacking–Keating). The Weinstein mirror is $X = \{(u, v) \mid uv \neq 1\}$.

**Corollary**

Let $K, L \in \mathcal{W}(X)$ and let $\mathcal{F}_L, \mathcal{F}_K \in D^{b}_{\text{coh}}(U)$ be the mirror sheaves. Then

$$\frac{\log \Gamma^{\text{ham}}_{K,L}}{\log n} = \frac{\log \Gamma_{K,L}}{\log n} = \dim(\text{supp} \mathcal{F}_K \cap \text{supp} \mathcal{F}_L).$$

(16)

**Remark**

The proof of this corollary uses the fact that $D$ is ample. In particular, it immediately generalizes to higher dimensions if one has HMS for pairs.
Relation to entropy

Let $B$ be a smooth and proper (idempotent complete, pre-triangulated) category and let $S : B \to B$ be an endofunctor. The entropy of $S$ is

$$\limsup_n \frac{1}{n} \log \dim \text{Ext}^* (G, F^n G)$$

for $G$ any split generator (Dimitrov–Haiden–Kontsevich–Katzarkov). There is also a notion of slow entropy: $\limsup_n \frac{1}{\log n} \log \dim \text{Ext}^* (G, F^n G)$ (Fan–Fu–Ouchi).

Suppose $\phi : M \to M$ is a diffeomorphism; let

$$\sup_{N \subset M} \limsup_n \frac{\log \text{Vol}_g (\phi (N))}{n}$$

be the volume growth (this agrees with the topological entropy by results of Yomdin and Newhouse). Similarly, we can consider the slow volume growth

$$\sup_{N \subset M} \limsup_n \frac{\log \text{Vol}_g (\phi (N))}{\log n}$$

The growth of $\Gamma_{K,L}$ gives lower bounds on these various entropies. This uses work of Frauenfelder–Schlenk, Alves–Meiwes, Sylvan, and others.
Questions
Other categorical compactifications I

- Instead of working with $\mathcal{W}(-)$, we can work with categories of microlocal sheaves (this is essentially equivalent by work of Ganatra–Pardon–Shende);
- In some cases, if $Y$ is a singular variety and $f : X \to Y$ is a resolution, then $f_* : D^{b}_{\text{coh}}(X) \to D^{b}_{\text{coh}}(Y)$ is a smooth categorical compactification. We unfortunately don’t have any good computations in this setting.
- We can consider smooth categorical compactifications of wrapped Fukaya categories of Liouville sectors or (equivalently) Liouville pairs.

Example

Suppose that $\ell \subset S^3$ is a knot and let $\Lambda \subset S^* S^3$ be the conormal of $\ell$. Then we can embed $\Lambda$ into the conormal of a triangulation of $S^3$; this gives a smooth categorical compactification of $\mathcal{W}(T^* S^3, \Lambda)$.

If $L \subset T^* S^3$ is a cotangent fiber (with $\partial_{\infty} L \cap \Lambda = \emptyset$), what does $\Gamma_{L,L}$ remember about $\ell$? It is related to other knot invariants?
At the moment, we only know how to construct smooth categorical compactifications for $\mathcal{W}(X, c)$ if $X = T^*M$.

**Question**

*Suppose that $X$ is a Liouville sector? Does $X$ always admit a compactifying stop? (i.e. a mostly Legendrian (up to deformation) stop $f$ such that $\mathcal{W}(X, f) \rightarrow \mathcal{W}(X)$ is a smooth categorical compactification).*

If $X$ is a Liouville manifold, we saw that this follows from Giroux’s work.
Let $X$ be Weinstein and let $K, L$ be Lagrangian submanifolds (cylindrical at infinity).

**Question**

1. *if* $\Gamma_{K,L}$ *grows faster than a polynomial, does this imply that it grows exponentially?*

2. *if* $\Gamma_{K,L}$ *grows slower than a polynomial, does this mean that*

   $$\lim \log \Gamma_{K,L} / \log n \in \mathbb{Z}$$  \hspace{1cm} (19)

   *(i.e. the growth rate is integral).*
Integrality II

Here is some (extremely slim) evidence:

1. If $X = T^* M$, then $\text{Perf } \mathcal{W}(X) = \text{Perf } C_{-\ast}(\Omega M)$. Now there is a (somewhat) related dichotomy in rational homotopy theory which states that the growth of $\sum_{i \neq n} \dim H_i(\Omega M)$ as a function of $n$ is either polynomial or exponential.

2. If $X$ is mirror to the complement of an anticanonical divisor, then this follows from the previous slide.

Remark

You can also ask similar questions about entropy/polynomial entropy (e.g. is the entropy of the “wrap-once functor" on a Weinstein pair either polynomial or exponential; is the volume growth rate of a Reeb flow either polynomial or exponential, etc.)
Thank you very much for your attention!