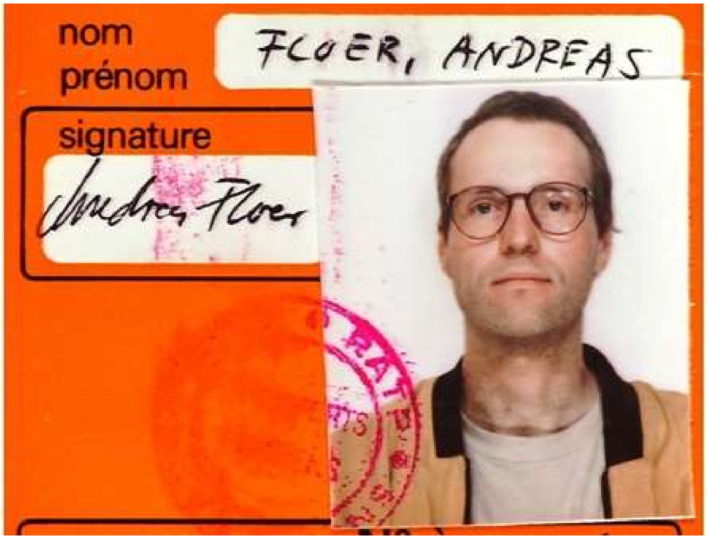
The Floer Jungle

35 Years of Floer Theory



Andreas Floer 23 August 1956 – 15 May 1991

My talked is based in part on a book project about Floer with

Siobhan Roberts Author, Science Journalist and Senior Editor on Computing of the MIT Technology Review
(NYT, The New Yorker, Quanta, Smithsonian,…)

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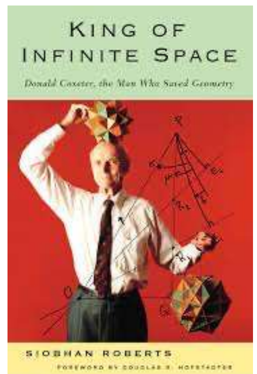

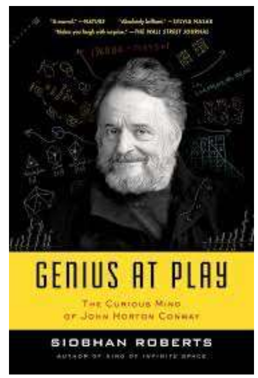

Many thanks to **Rolf Kaiser** who knew Andreas Floer since around 1976 and has been an indispensable source for us and was able to connect us with sources who were not on any of the mathematicians’ radar screens.

Atiyah about Floer

Sir Michael Atiyah 1929-2019

The first time I met Grothendieck, I remember he was a whirlwind,
he talked for hours nonstop—that was an experience never to be forgotten.
Floer was the other end of the spectrum— there were more reasons to forget:
he was a quiet person, subdued, he didn't make a big impact.
As for Floer's theory, I was keen to spread its impact around.
It was Fields Medal material, definitely; it was up to the Fields' standard.
I did a lot of propaganda at conferences in those days.
My main role was trying to explain new developments of problems.
People thought I kept bad company because I mixed with physicists— physicists were too woolly.
But Floer had spectacular new ideas that they could use, and at just the right moment.
He bridged a gap—he provided stepping stones—between mathematics and physics.
Mathematics comes in fits and starts and here was a brand new idea that came from nowhere,
or seemed to, and opened a door to the future, a period of heightened activity.
And that's rare. Donaldson, Floer, Witten, their ideas were inspired by each other –
they opened the door and went out and explored all the wonderful new trees and flowers.

—Michael Atiyah

Andreas Floer was a very complex person.

His life was organized in several “parallel universes”, for example

MATH **MUSIC** …………. …………

He belonged to each such universe, but the other inhabitants with few exceptions did not know about the other ones and their inhabitants.

For example in the **MUSIC** universe a few had heard about Misha Gromov, Alan Weinstein and Eduard Zehnder. This was a clear sign that Andreas Floer cared about them even if he wasn’t able to show it.

During one single day these universes intersected!

Andreas Floer's mathematical contributions come in one powerful burst during a period of about five years.

Floer, A. ; Zehnder, E. Fixed point results for symplectic maps related to the Arnol'd conjecture. Dynamical systems and bifurcations (Groningen, 1984), 47--63, Lecture Notes in Math., 1125, Springer, Berlin, 1985.

Floer, Andreas. Proof of the Arnol'd conjecture for surfaces and generalizations to certain Kähler manifolds. [Jan 1985, March 1986]
Duke Math. J. 53 (1986), no. 1, 1--32.

Floer, Andreas . A refinement of the Conley index and an application to the stability of hyperbolic invariant sets.
Ergodic Theory Dynam. Systems 7 (1987), no. 1, 93--103. [March 1985, March 1987]

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Floer, A; Weinstein, A. Nonspreading Wave Packets for the Cubic Schroedinger Equation with a Potential. [June 85, Dec 1986]
J. Funct. Anal. 69, 397-408 (1986)

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Andreas Floer, Monopoles on Asymptotically Euclidean 3-Manifolds, Bull. AMS, Vol 16, No 1 (1987) [July 1986, Jan 1987]

4. A. Floer, *The configuration space of Yang-Mills-Higgs theory on asymptotically flat manifolds*, Comm. Math. Phys. (to appear).

5. ____, *Monopoles on asymptotically flat manifolds*, Comm. Math. Phys. (to appear).

Floer, A. The configuration space of Yang-Mills-Higgs theory of asymptotically flat manifolds. The Floer memorial volume, 43--75, Progr. Math., 133, Birkhäuser, Basel, 1995.

Floer, A. Monopoles on asymptotically flat manifolds. The Floer memorial volume, 3--41, Progr. Math., 133, Birkhäuser, Basel, 1995.

Floer, Andreas. Morse theory for fixed points of symplectic diffeomorphisms.
Bulletin (New Series) of the American Mathematical Society Vol. 16, Issue 2 (1987), 279-281 [August 1986, April 1987]

The author wishes to thank M. Gromov, A. Weinstein, and E. Zehnder for valuable discussions.

Floer, Andreas . A relative Morse index for the symplectic action. Comm. Pure Appl. Math. 41 (1988), no. 4, 393--407. [March 1987, June 1988]
Floer, Andreas . The unregularized gradient flow of the symplectic action. Comm. Pure Appl. Math. 41 (1988), no. 6, 775--813. [June 1987, Sept 1988]
Floer, Andreas . Morse theory for Lagrangian intersections. J. Differential Geom. 28 (1988), no. 3, 513--547. [May 1987, January 1988]
Floer, Andreas . Witten's complex and infinite-dimensional Morse theory. J. Differential Geom. 30 (1989), no. 1, 207--221. [Jan 1988, Jan 1989]
Floer, Andreas . Cuplength estimates on Lagrangian intersections. Comm. Pure Appl. Math. 42 (1989), no. 4, 335--356. [August 1987, June 1989]
Floer, Andreas . Symplectic fixed points and holomorphic spheres. Comm. Math. Phys. 120 (1989), no. 4, 575--611. [May 1988, Jan 1989]

Floer, Andreas . An instanton-invariant for 3-manifolds. Comm. Math. Phys. 118 (1988), no. 2, 215--240. [May 1988, June 1988]
Floer, Andreas. Instanton homology and Dehn surgery. The Floer memorial volume, 77--97, Progr. Math., 133, Birkhäuser, Basel, 1995

Floer, A.; Hofer, H.; Viterbo, C. The Weinstein conjecture in P×Cl. Math. Z. 203 (1990), no. 3, 469–482. [Nov 1988, Jan 1990]

Other Posthumous Publications

Floer, A. ; Hofer, H. Coherent orientations for periodic orbit problems in symplectic geometry. Math. Z. 212 (1993), no. 1, 13--38.

Floer, Andreas ; Hofer, Helmut ; Salamon, Dietmar . Transversality in elliptic Morse theory for the symplectic action.
Duke Math. J. 80 (1995), no. 1, 251--292.

Research work done in Paris during Summer 88 and talks at MSRI during August/September 88
Talks at MSRI according Alan Weinstein's notes.

HH (8/29/1988) Symplectic Capacities I
HH (8/30/1988) Symplectic Capacities II
AF (9/2/1988) Capacities via Elliptic Morse Theory
HH (????) Symplectic Homology

Floer, A. ; Hofer, H. Symplectic homology. I. Open sets in C^n. Math. Z. 215 (1994), no. 1, 37--88.

Floer, A. ; Hofer, H. ; Wysocki, K. Applications of symplectic homology. I. Math. Z. 217 (1994), no. 4, 577--606.

Cieliebak, K. ; Floer, A. ; Hofer, H. Symplectic homology. II. A general construction. Math. Z. 218 (1995), no. 1, 103--122.

Cieliebak, K. ; Floer, A. ; Hofer, H. ; Wysocki, K. Applications of symplectic homology. II. Stability of the action spectrum.
Math. Z. 223 (1996), no. 1, 27--45

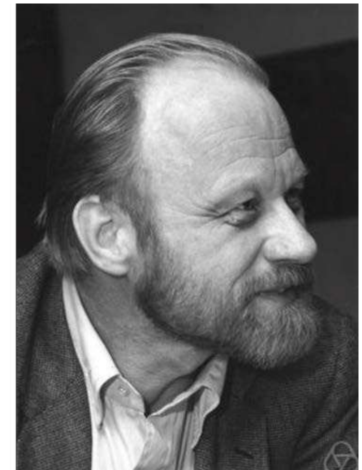
The Mathematical Landscape

One could try to base the solution of this boundary value problem on the well known variational principle which calls for the extrema of the functional

$$(1.5) \qquad S = \int \sum y_k \, dx_k - \lambda(H - c)\, ds = \int_0^1 \left(\left(y, \frac{dx}{ds} \right) - \lambda(H - c) \right) ds$$

over the class of periodic vector functions $z(s) = (x(s), y(s))$ of period 1 and the parameters λ. The Euler equations for this variational principle are given precisely by (1.2) together with $H = c$.

However, this variational principle is very degenerate, for example even the Legendre condition is violated, and is certainly not suitable for an existence proof. To overcome this difficulty we solve the above boundary



J. Moser, Periodic orbits near an equilibrium and a theorem by Alan Weinstein. COMMUNICATIONS ON PURE AND APPLIED MATHEMATICS, VOL XXIX (1976)

Moser was a scholar with very broad knowledge.
Moser was very precise.

And in this case he was wrong!

Why did Moser say this?
What he said was based on state of the art knowledge about so called critical point theory, i.e. Morse Theory.

Moser paraphrased:
These type of functionals are pointless to study since there is no relationship between the structure of critical points and the topology of the ambient space!

$$S = \int \sum y_k \, dx_k - \lambda(H-c)\, ds = \int_0^1 \left(\left(y, \frac{dx}{ds} \right) - \lambda(H-c) \right) ds$$

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knowledge about so called critical point theory,
i.e. Morse Theory.

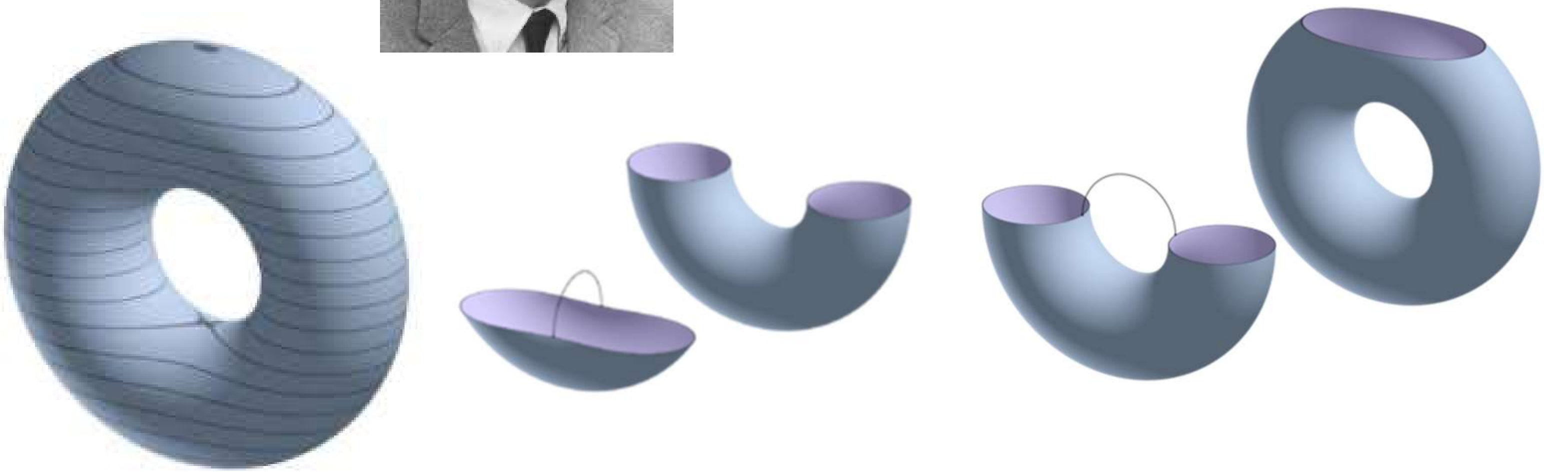
Moser paraphrased again:

There can never be a Floer Theory!

Marston Morse

1892-1977

Morse Theory



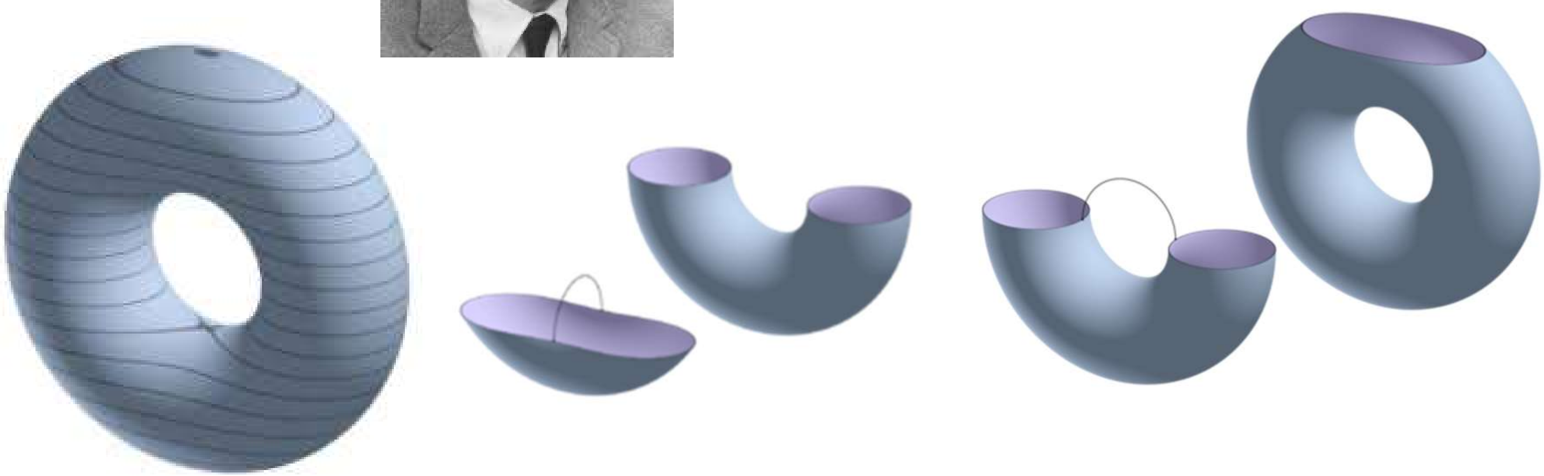
Analytical tool to study a smooth manifold is the negative gradient flow

$$\dot{z} = -\nabla h(z)$$

Marston Morse

Morse Theory

1892-1977



$h : M \to \mathbb{R}$ with non-degenerate critical points and assume $h^{-1}(d)$ contains for every $d \in \mathbb{R}$ at most one critical point. Denote by $m(x)$ the Morse-index of a critical point with $h(x) = d$ and define $M^d = h^{-1}((-\infty, d])$ and $\dot{M}^d$ is $M^d \setminus \{x\}$.

$$0 \to H_{m(d)}(\dot{M}^d) \to H_{m(d)}(M^d) \to \mathbb{Z} \to H_{m(d)-1}(\dot{M}^d) \to H_{m(d)-1}(M^d) \to 0.$$

Moser’s Worry

In infinite dimensions for our functional the Morse index are infinite in both direction.

Terrible things can happen!

Let $\bar{B}$ and S be the closed unit ball and unit sphere in a separable Hilbert space of infinite dimensions. Then

- $S \subset \bar{B}$ is a strong deformation retract.
- S is a contractible topological space

Algebraic Topology in the context of our functional is going up in flames!

BAD!

In Moser's case $\mathrm{Loop} \to \mathbb{R}$ with $L = \mathrm{Loop}$

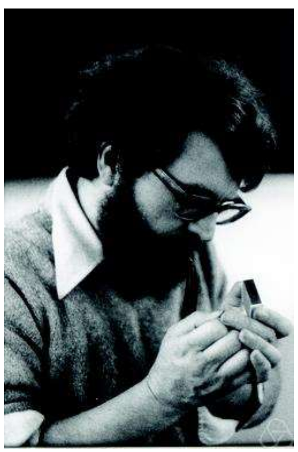
$$0 \to H_*(\dot{L}^d) \to H_*(L^d) \to 0.$$

Homology does not change!

Paul Rabinowitz to the rescue

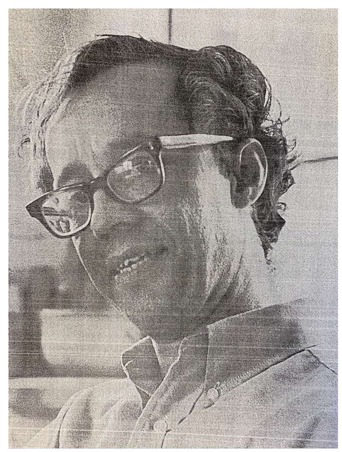
1978-79 ====➔ Weinstein Conjecture

Frank Clarke

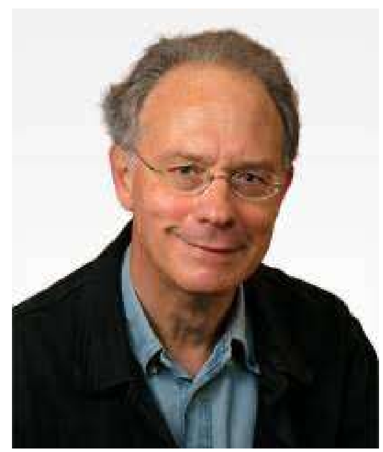


Edi Zehnder

Sabbatical Fall 1982
In Bochum



Charley Conley
(1933-1984)



Ivar Ekeland

=>

Claude Viterbo

- Wave equation Problem (1978)
  Here Moser understands that he was wrong and points Rabinowitz's attention to finite-dimensional Hamiltonian systems.

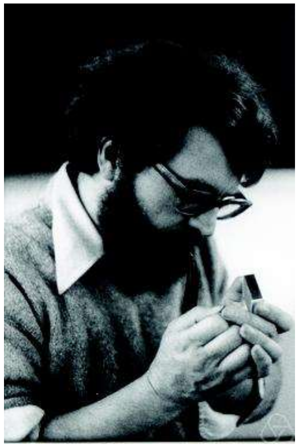
- Rabinowitz result on the existence of periodic orbits on star-shaped Hamiltonian energy surfaces (1978)
- A second result in this direction by Rabinowitz (1979)

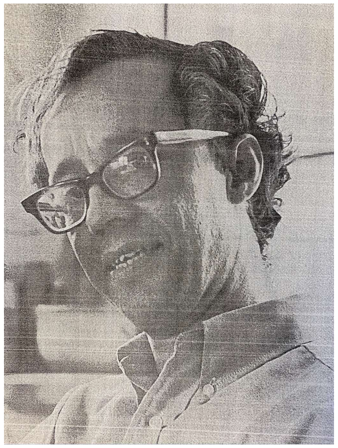
(Weinstein was the referee; lead to the Weinstein conjecture in 1979)

- Ekeland's Morse Theory for convex Hamiltonian systems (1984)
- Viterbo Conjecture (2000)

FIM – Institute for Mathematical Research 1982, Zuerich

We keep these three guys in mind and have a look at what Floer is up to.





John Mather
(1942-2017)

Visitors at the FIM

Conley/Zehnder for a few weeks
Before Christmas

WHERE IS ANDREAS FLOER ?

Floer has just finished his undergraduate studies in Bochum in 1982.

Starts as a graduate student in Berkeley in Fall 1982
He must have applied in January 1982 to be admitted to graduate school in Berkeley.

Why did he decide to go to Berkeley?

A friend of his said: “Andreas wanted to see the world and wanted to go to a place with a lot of mathematical research.”

Andreas in Berkeley as a graduate student

HH asked Weinstein: Alan, who was his advisor at Berkeley?
Alan: ”He was working with Taubes”

HH then asked Taubes: “Cliff, Alan told me that you were Floer’s advisor”

Cliff: “I thought he was working with Alan”

Cliff joins Berkeley faculty in Fall 1983 and then becomes Harvard faculty in June 1985

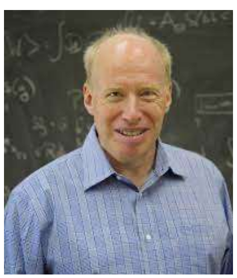
Graduate student mentoring at Berkeley at its finest:
The graduate student you never knew you had.

Andreas Floer was “pflegeleicht”, i.e. “easy to care for”.

Karen Uhlenbeck

M. Freedman (1973)

Karen Uhlenbeck and Michael Freedman run a seminar on Donaldson's work in Fall 1982 and this is the first MSRI seminar.
(According to Freed: "This was really the first MSRI event")



Floer is a participant.

Karen remembers: "Floer gave one of the seminars, and Dan and I had to fix up his proof (he wasn't on top of everything)"

Dan Freed (designated notetaker)

Floer learns in Berkeley about:
Bubbling-off (Uhlenbeck)
Gluing (Taubes)
Yang-Mills equations

Floer reads Witten's paper on Supersymmetry and Morse theory.
He will be in 1983 influenced by what is happening at the FIM.

remember  ?

The background from his Diploma thesis is algebraic topology.

1976

The Floer Jungle

1976

The Floer Jungle

Prologue of my book with Siobhan Roberts

We're embarking on this safari to explore the work of the German geometer Andreas Floer.

One approach to mathematics—a path rarely if ever taken by Floer—is to imagine that you're standing in front of a big wall (the problem) and you have to get through the wall by any means necessary.
Maybe a sledgehammer will work; more likely, however, some heavy artillery will be necessary.
But really, it's you against the wall. You run up against it, and run and run and run. The wall never moves.
If you're powerful enough, gradually you might create a crack. You run against it some more.
The crack gets bigger. Eventually you might get through. That's a success story. Your head hurts.
But the adrenalin overrides the pain.

Another strategy—Floer's preferred path—is to imagine you are an anthropologist in a mathematical jungle.
Understanding the mathematics, the creatures under investigation, is like exploring the jungle's wildlife and vegetation.
Out on a fieldwork expedition, you set off every morning from camp with your machete and clear encroaching brush and dangling vines as you go. You're working away, moving forward. Sometimes it's a slog and you spend too much time following a boring and seemingly trivial tributary, or, worse, you get mired in mud.
Other times you happen upon lush valleys and wide vistas. Maybe you cross a ditch, and an hour later you cross another ditch, and after crossing a bunch of ditches you realize that all these ditches are part of a huge pattern that some ancient civilization dug into the earth, like the Nazca Lines.

Unlike the problem-oriented approach, in the jungle you're not fixated on any one creature.
Instead of running against the wall, you veer to the right and wander left, through a woods and around a corner, and there you find a door, unlocked. It opens onto the same place: the other side of the wall.
Not only have you achieved what you wanted and arrived at your desired destination, but you've also learned about lots of unexpected creatures along the way.

Welcome to *The Floer Jungle…*

Once Upon a Time in Russia

Vladimir Arnold
(1937-2010)

A Remarkable Conjecture

In his book on classical mechanics

Theorem: Every symplectic diffeomorphism of a compact symplectic manifold, homologous to the identity, has at least as many fixed points as a smooth function on this manifold has critical points (at least if this diffeomorphism is not too far from the identity).

At other places one finds a version just for two-torus

Arnold Conjectures about symplectic fixed points in the 60's

1978 Eliashberg finds a solution for the Arnold conjecture in the case of surfaces. Before the paper was complete, he gives a preliminary draft to Anatole Katok, who smuggles it out from Russia, which he delivers to Gromov.

It turns out the preliminary version had a gap but Eliashberg was able to send Gromov a second draft.

Gromov gives the(?) paper to a group of French mathematicians and they get stuck trying to understand the two-torus-case.

Yasha Eliashberg

Anatole Katok
(1944-2018)

Misha Gromov

HH: More than 10 years ago I asked Gromov and he thought that there was the possibility that he forwarded the draft with the gap to the French mathematicians who ran a seminar about it……

Francois Laudenbach to HH: “I should tell you that there was a tradition of active working group and the use was that almost everybody in the Topology group at Orsay was coming, including Chenciner, Herman, Marin, Fathi, Bonahon, Audin, Cerf, Rosenberg, Poenaru, Douady ...”

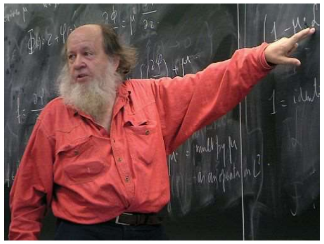
Michele Audin

Daniel Bennequin

Francois Laudenbach

Michel Herman (1942-2000)

Albert Fathi



Adrien Douady (1935-2006)

In Fall 82 John Mather visits Michel Herman and learns about the failed attempt and the Arnold conjecture.

After he returns to Zurich, Conley and Zehnder tell him what they are doing, and Mather tells them with their methods they might have a shot at the Arnold Conjecture for the two-torus. They are able to do T^{2n} within a few days.

Already a few weeks later Zehnder gives a colloquium talk at the University of Zurich about it. The news travels to Paris and the attempts to understand Eliashberg’s paper cease.

Back to Andreas Floer

Floer spends the summers 1983 and 1984 in Bochum and his “Ersatzdienst” (instead of military service) was at least one of the reasons.

At the latest he learned about the Conley-Zehnder result in Summer 1983 from Zehnder directly and working on the surface case of the Arnold conjecture was a natural thing to do.

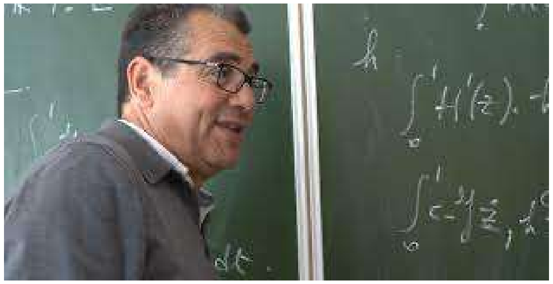
Floer has this achieved rather fast and shelves the material away and he attempts more general cases and that is where he gets stuck.

We are now at the beginning of Summer 1984.

The only way to get unstuck is by modifying Gromov’s methods, but the paper on “Pseudoholomorphic Curve Theory” is submitted to Inventiones in January 1985 and only at that time a preprint is only floating around.

Floer quits Berkeley (why?) and obtains his Ph.D. in Bochum in December 1984 with the previously shelved away material on surfaces and then he goes back to Berkeley.

Here is an explanation according to



Abbas Bahri
(1955-2016)

Bahri (U. Chicago at that time) had met Floer and had told him that HH can prove the Arnold conjecture in general.

That was true, but it was a different one!

So it is possible that Floer “panicked”.

Beginning of 1985 Floer is back in Berkeley and it seems that during the period January 85- May 85 Floer develops his major ideas.

January 85

Algebraic topology background, Bubbling-off, Gluing, Yang-Mills equations, Conley-Zehnder paper, Witten's paper on Supersymmetry and Morse theory.

February 85 – April 85

Alan Weinstein runs a seminar on Gromov's pseudoholomorphic curves paper and the other speakers are Andreas Floer, Tom Mrowka and Chuu-Lian Terng.

June 85

According to Taubes: Floer visits him in his office and says something along the lines: “I can do infinite-dimensional Morse Theory”.

Also according to Taubes: He wasn't too impressed at the time.

In my interpretation it means that Andreas Floer has an idea in the direction of the Arnold Conjecture in more general cases.

Floer starts in Stony Brook in 1985 as a postdoc/instructor and………………

He has a plan and he is working!

It is a bold idea to bring all these things together!

Floer gives a talk at the Arbeitstagung in Bonn on June 15th 1986.

This seems to be his first lecture on what will become Floer Theory.

There Atiyah meets him probably for the first time.

Programm der Mathematischen Arbeitstagung 1986 (I)

Freitag, den 13.6.1986

16.00 - 17.00 Uhr	M.F. ATIYAH: The logarithm of the Dedekind η-function

Samstag, den 14.6.1986

10.00 - 11.00 Uhr	CH. SOULÉ: Higher dimensional Arakelov theory
11.45 - 12.45 Uhr	M. KRECK: 7-dimensional Einstein manifolds with SU(3) x SU(2) x U(1) - symmetry
17.00 - 18.00 Uhr	K. FUKAYA: Collapsing and eigenvalues

Sonntag, den 15.6.1986

9.45 - 10.00 Uhr	Festlegung der nächsten Vorträge
10.00 - 11.00 Uhr	A. FLOER: Holomorphic curves and fixed points of symplectic maps
11.45 - 12.45 Uhr	P. KRONHEIMER: Gravitational instantons and Kleinian singularities
17.00 - 18.00 Uhr	G. FALTINGS: Hodge-Tate structures

The pressure on Floer after this talk is enormous!

A Time-Line from the Floer Perspective

Flow based	Morse (~1930):	Morse-Theory using gradient flow and algebraic topology
	Palais and Smale (~1960):	Palais-Smale condition and finite Morse indices allow generalization to infinite dimensions
	Rabinowitz (1978):	Palais-Smale condition and arbitrary Morse indices using some kind of Fredholm structure for the gradient flow allows to study the action functional in certain cases, serious technical limitations.

Certain gradient Lines

Witten (1983) rediscovers what was known to Milnor and Thom, but never received too much attention, that one can do the algebraic topological side of Morse-Theory just knowing gradient lines between critical points of Morse index difference 1 and 2 (in the generic situation)

Conley-Zehnder Theorem (1983)

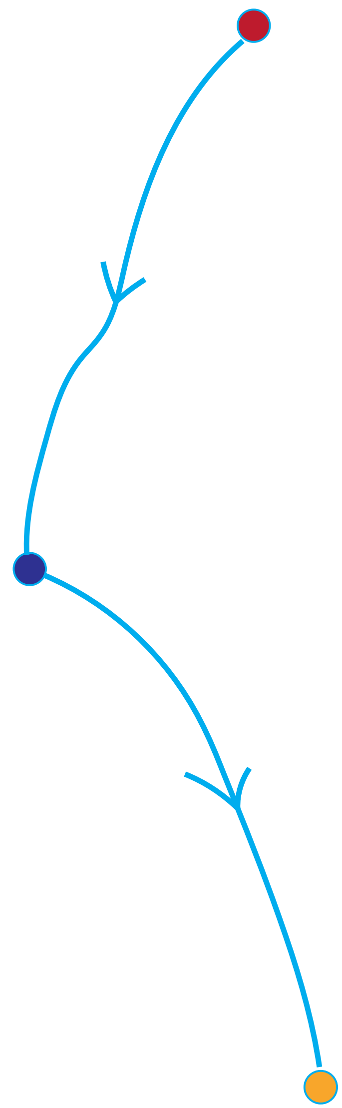
Advance of technology

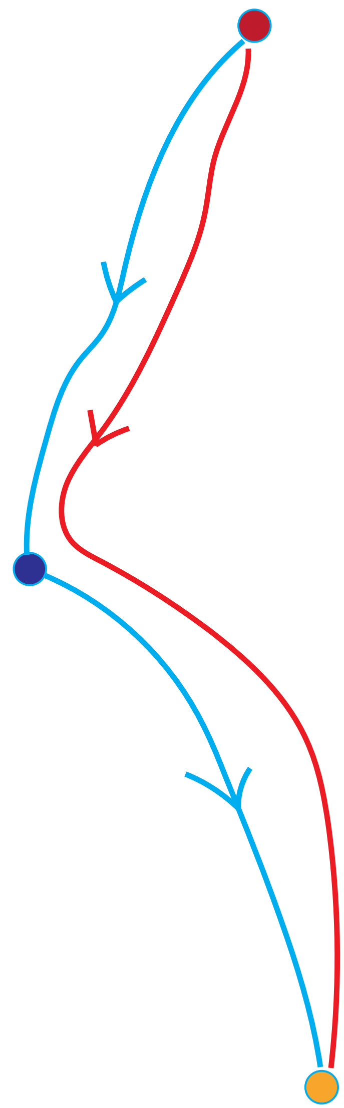
Gromov (1985) Pseudoholomorphic curve theory

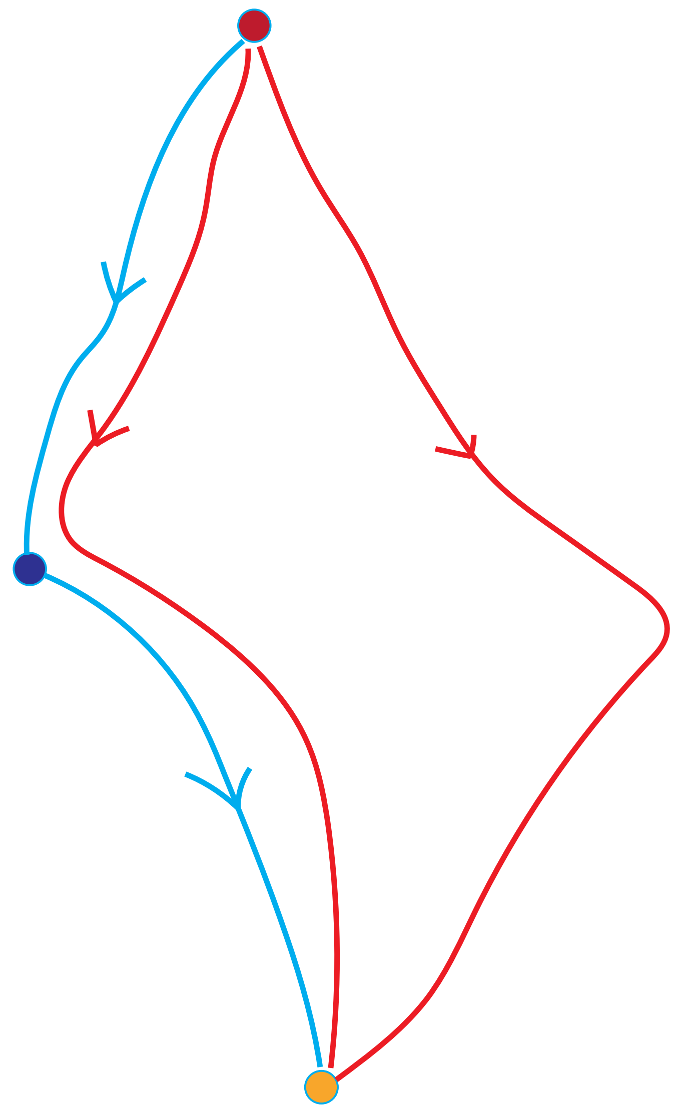
PDE based

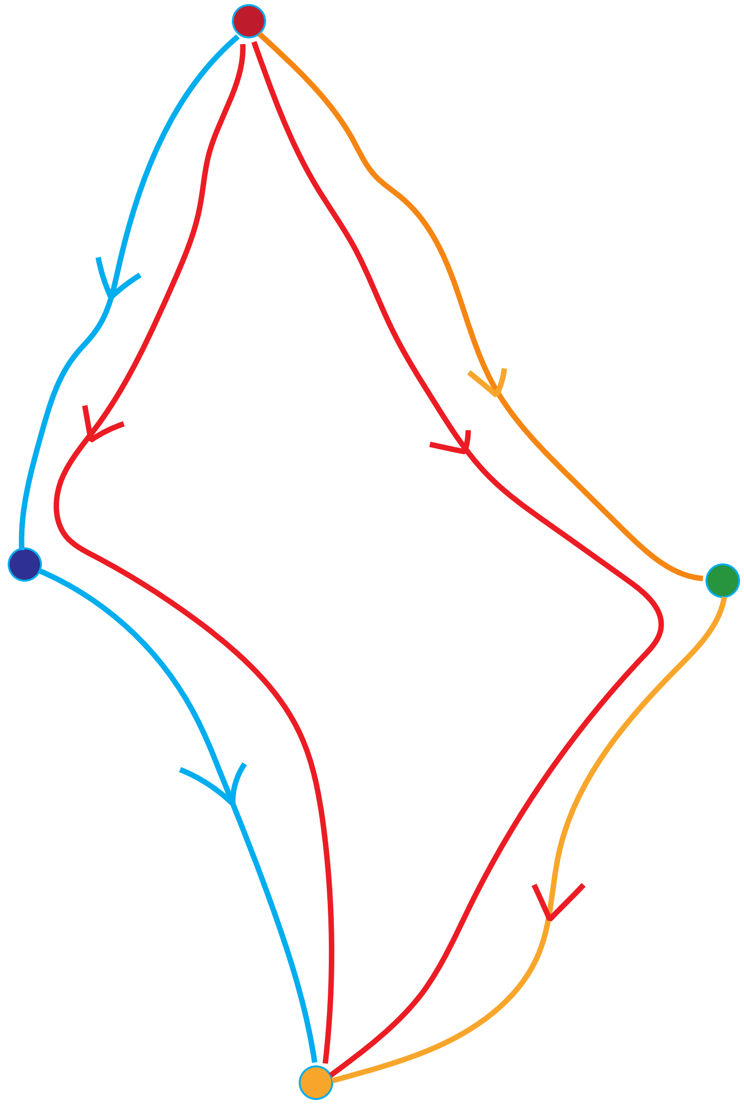
Floer (1986--): We just need a system of interrelated PDE's with enough compactness properties, and a suitable notion of a finite difference of two perhaps infinite Morse indices to have some theory.

Floer-theoretic methods which generalize Morse theory but also allow many novel constructions.

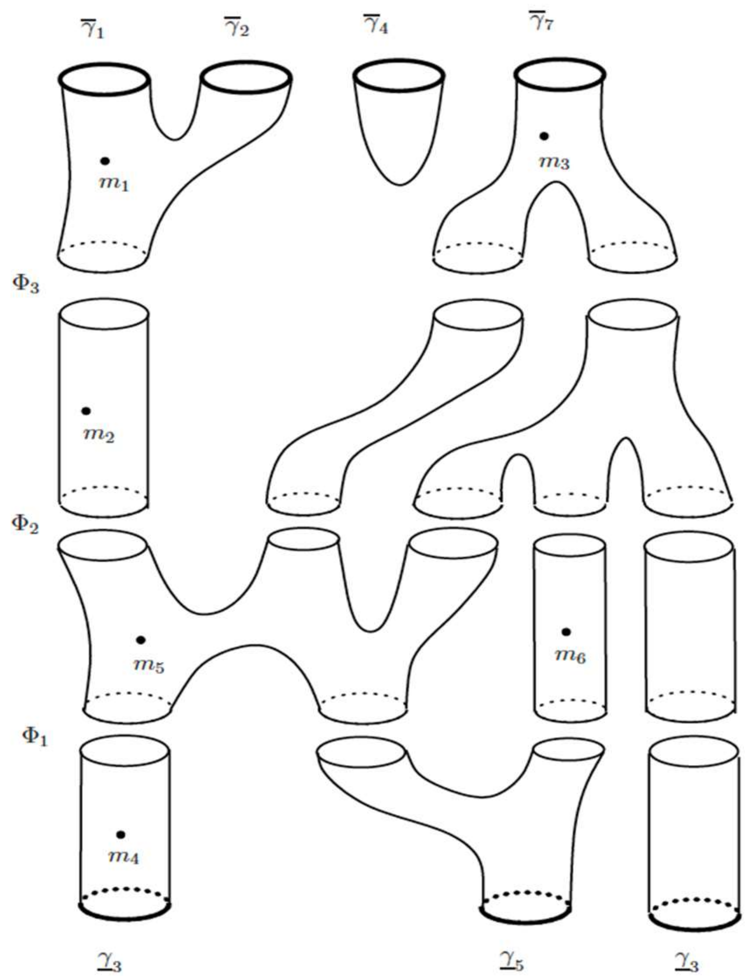








Today we study even more complicated situations which lead to algebraically much more complicated objects.



How was the reception of Floer’s idea, particularly before the ideas were in written form?

Quite Mixed!

On the occasion of receiving the 1997 Steele Prize for the “Pseudoholomorphic Curve Paper” you find in Gromov’s response the following, were he talks about pseudoholomorphic curves:

“**Floer has morsified them by breaking the symmetry, and I still cannot forgive him for this. (Alas, prejudice does not pay in science.)** McDuff started the systematic hunt for them which goes on till present day. And what goes on today goes beyond these lines and the pen behind them.”

Dusa and “collaborator” Thomas in 1985

“**Floer has morsified them by breaking the symmetry, and I still cannot forgive him for this. (Alas, prejudice does not pay in science.)**”

Il Ciocco,
October 1986

NATO CONFERENCE, one of the organizers is Rabinowitz

Zehnder explains Floer’s theory based on what Floer told him and Gromov’s thinks the ideas are deeply flawed (and, of course, changes his mind later).

The floor is now open for the discussion!

Many thanks for listening.