Cylindrical contact homology of links of simple singularities

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Outline

1. Overview of simple singularities and their links as Seifert fiber spaces
2. Computation of cylindrical contact homology using Morse functions invariant under symmetry groups
3. Realization of Seifert fiber structure by the homology groups
A **simple singularity** of a complex 2-dimensional variety may be locally modeled by $\mathbb{C}^2/G$ for a finite subgroup $G \subset SU(2)$.

- McKay correspondence $\implies \mathbb{C}^2/G \cong V := V(f)$, for some complex polynomial $f \in \mathbb{C}[x, y, z]$

- The **link** of the singularity, $L$, is defined as $S^5_\xi(0) \cap V$ and has a natural contact structure, $\xi_L$

- There is a contactomorphism $(L, \xi_L) \cong (S^3/G, \xi)$, where $\xi$ is the descent of the standard contact structure on $S^3$ under the quotient
Consider the double cover of Lie groups $P : SU(2) \rightarrow SO(3)$. Let $H := P(G)$.

By the classification of finite subgroups of $SU(2)$, $G$ is either
- **Cyclic**, and $H$ is also cyclic,
- **Binary dihedral** $\mathbb{D}_{2n}^*$, and $H = \mathbb{D}_{2n}$, or
- **Binary polyhedral** $\mathbb{T}^*$, $\mathbb{O}^*$, or $\mathbb{I}^*$, and $H$ is $\mathbb{T}$, $\mathbb{O}$, or $\mathbb{I}$.

The $G$-action on $S^3$ is fixed point free, unlike the $H$-action on $S^2$.

The quotient $S^2/H$ is an **orbifold**, homeomorphic to $S^2$. 
$S^3 / G \rightarrow S^2 / H$ as a Seifert fiber space

An $H$-invariant Morse function $f : S^2 \rightarrow \mathbb{R}$ provides a descent to an orbifold Morse function $f_H : S^2 / H \rightarrow \mathbb{R}$.

For $\varepsilon > 0$ small, define $\lambda_\varepsilon := (1 + \varepsilon p^*f_H)\lambda$, the perturbed contact form.

The fiber over an orbifold point $p \in S^2 / H$ is an exceptional fiber, denoted $\gamma_p$. 
### Lemma

The Reeb vector field of the **perturbed contact form**, $\lambda_\varepsilon = (1 + \varepsilon p^* f_H) \lambda$, is given by

$$R_\varepsilon = \frac{R}{1 + \varepsilon p^* f_H} - \varepsilon \frac{\tilde{X}_f}{(1 + \varepsilon p^* f_H)^2}.$$ 

Here, $\tilde{X}_f$ is a lift of the Hamiltonian vector field for $f_H$. We see that the $\gamma_p$ are embedded Reeb orbits of $\lambda_\varepsilon$.

### Proposition

**Given** $L > 0$, there exists an $\varepsilon_0 > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$, then all $\gamma \in \mathcal{P}^L(\lambda_\varepsilon)$ are nondegenerate, and project to orbifold critical points of $f_H$ under $p$.

We take $L_N \to \infty$, the corresponding $\varepsilon_N \to 0$, and we define $\lambda_N := \lambda_{\varepsilon_N}$, this is $L_N$-dynamically convex.
Example: $\mathbb{T}$-invariant Morse function on $S^2$

Let $f : S^2 \to \mathbb{R}$ be a $\mathbb{T}$-invariant Morse function on $S^2$ with $\text{Crit}(f) = \text{Fix}(\mathbb{T})$, its descent is denoted $f_\mathbb{T} : S^2/\mathbb{T} \to \mathbb{R}$. 

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Graded generators in the tetrahedral setting

We have three embedded Reeb orbits $V$, $E$, and $F$ in $S^3/T^*$ which project to the three orbifold points $v$, $e$, and $f$ of $S^2/T$.

<table>
<thead>
<tr>
<th>CZ − 1</th>
<th>Orbits</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>$V, V^2, V^3, E, F, F^2$</td>
</tr>
<tr>
<td>1</td>
<td>$E^2$</td>
</tr>
<tr>
<td>2</td>
<td>$V^4, V^5, V^6, E^3, F^3, F^4, F^5$</td>
</tr>
<tr>
<td>3</td>
<td>$E^4$</td>
</tr>
<tr>
<td>4</td>
<td>$V^7, V^8, V^9, E^5, F^6, F^7, F^8$</td>
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<tr>
<td>...</td>
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These orbits and their gradings are used to compute the action filtered cylindrical contact homology of $S^3/T^*$.
The cylindrical contact homology

**Theorem (D, 2021)**

Let $G \subset SU(2)$ be a finite nontrivial group, and let $m \in \mathbb{N}$ denote $|\text{Conj}(G)|$, the number of conjugacy classes of $G$. Then

$$\lim_{N \to \infty} CH^L_N(S^3 / G, \lambda_N, J_N) \cong \bigoplus_{i \geq 0} \mathbb{Q}^{m-2}[2i] \oplus \bigoplus_{i \geq 0} H_*(S^2)[2i].$$

- Here, $L_N \to \infty$, $\lambda_N$ is $L_N$-dynamically convex, $J_N$ is a generic $\lambda_N$-compatible almost complex structure.
- Copies of $H_*(S^2)$ are to be interpreted as the **orbifold Morse homology** of the orbit space, which is an orbifold $S^2$.
- The dimension of the $\mathbb{Q}^{m-2}$ term is the total isotropy of the orbit space.
Tetrahedral example

Theorem (D, 2021)

\[
\lim_{N \to \infty} CH^*_{N}(S^3/\mathbb{T}^*, \lambda_N, J_N) \cong \bigoplus_{i \geq 0} \mathbb{Q}^5[2i] \bigoplus \bigoplus_{i \geq 0} H_* (S^2)[2i].
\]

\[G = \mathbb{T}^*, \quad m = 7, \quad m - 2 = 5\]
Realizing a contact McKay correspondence

- The McKay correspondence provides that $|H^*(Y_G; \mathbb{Q})| = m$, where $Y_G$ is the minimal resolution of the singularity $\mathbb{C}^2/G$.
- McLean and Ritter generalize this result for $G \subset SU(n)$, $Y_G$ a crepant resolution of $\mathbb{C}^n/G$.
- They prove this by relating the rank of $SH^*_+(Y_G)$ to the number of conjugacy classes of $G$.

Consider that our direct limit may be rewritten as

$$\lim_{\rightarrow} CH_{\ast}^{L_\mathcal{N}}(S^3/G, \lambda_\mathcal{N}, J_\mathcal{N}) \cong \begin{cases} \mathbb{Q}^{m-1} & \ast = 0, \\ \mathbb{Q}^m & \ast \geq 2 \text{ and even,} \\ 0 & \text{else.} \end{cases}$$

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Thank you for your time.
Comparing differentials

Definition (Eliashberg, Givental, Hofer)

Suppose $\mathcal{M}_1^J(\gamma_+, \gamma_-)/\mathbb{R}$ is a finite set for all $\gamma_\pm \in \mathcal{P}_{\text{good}}(\lambda)$. Define $\partial : CC_*(Y, \lambda, J) \rightarrow CC_*-1(Y, \lambda, J)$ by

$$\langle \partial \gamma_+, \gamma_- \rangle := \sum_{u \in \mathcal{M}_1^J(\gamma_+, \gamma_-)/\mathbb{R}} \epsilon(u) \frac{m(\gamma_+)}{m(u)}$$

- Here, $\epsilon(u) \in \{\pm 1\}$ is determined by a coherent choice of orientations, defined for cylinders between good Reeb orbits,
- $m(\gamma_+)$ and $m(u)$ are the covering multiplicities of $\gamma_+$ and $u$.

Definition (Cho, Hong)

The differential $\partial^{\text{orb}}$ of orbifold Morse homology is

$$\langle \partial^{\text{orb}} p, q \rangle := \sum_{x \in \mathcal{M}(p,q)} \epsilon(x) \frac{|\Gamma_p|}{|\Gamma_x|}$$
Technical considerations (1)

- Direct limits are over homomorphisms between filtered groups induced by **symplectic cobordisms**.
- The Fredholm index 0 cylinders under consideration are regular, by **automatic transversality**.
- **Compactness** of the moduli spaces holds due to strong relationships between action, CZ index, and free homotopy classes of Reeb orbits.
Proposition (D, 2021)

Suppose $\gamma_+ \text{ and } \gamma_- \text{ represent the same free homotopy class, where}$
\[ \gamma_+ \in \mathcal{P}^{L_N}(\lambda_N) \text{ and } \gamma_- \in \mathcal{P}^{L_M}(\lambda_M). \]

If $\mu_{CZ}(\gamma_+) = \mu_{CZ}(\gamma_-)$, then $m(\gamma_+) = m(\gamma_-)$ and $\gamma_\pm$ project to the same orbifold point of $S^2/H$.

If $\mu_{CZ}(\gamma_+) < \mu_{CZ}(\gamma_-)$, then $A(\gamma_+) < A(\gamma_-)$.

Importantly,

- The first bullet indicates that many of the moduli spaces $\mathcal{M}_0^J(\gamma_+, \gamma_-)$ are \textbf{empty}.

- The second bullet indicates that \textbf{there do not exist cylinders of negative Fredholm index} in our cobordisms.