Results on abundance of global surfaces of section

Umberto L. Hryniewicz (RWTH Aachen)

joint works with **A. Florio**, and with
V. Colin, **P. Dehornoy** and **A. Rechtman**

Symplectic Zoominar

CRM-Montréal, Princeton/IAS, Tel Aviv, and Paris

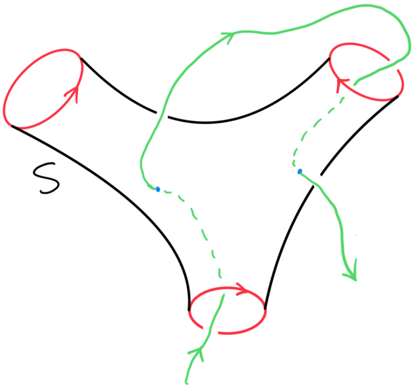
Global surfaces of section

M = closed 3-manifold
X = vector field on M
ϕ^t = flow of X

Definition. A *global surface of section* (GSS) for ϕ^t, or for X, is an embedded compact surface $S \subset M$ such that:

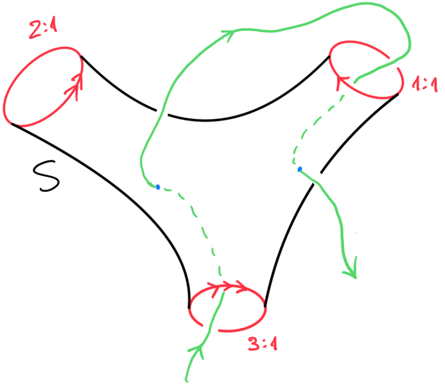
- ∂S consists of periodic orbits.
- X is transverse to $S \setminus \partial S$.
- $\forall p \in M \ \exists t_- < 0 < t_+$ such that $\phi^{t_\pm}(p) \in S$.

Global surfaces of section



Ghys: "Dynamicist's paradise"

Rational global surfaces of section (Birkhoff section)



Applications?

Applications I: Systolic Geometry on the two-sphere

A Riemannian two-sphere is pinched by $\delta \in (0, 1]$ if $K_{min} \geq \delta K_{max}$

ℓ_{min} = minimal length of a non-constant closed geodesic
ℓ_{max} = maximal length of an embedded closed geodesic

Theorem (Abbondandolo, Bramham, H., Salomão). If a Riemannian S^2 is pinched by more than $\frac{4+\sqrt{7}}{8} = 0.83\dots$ then

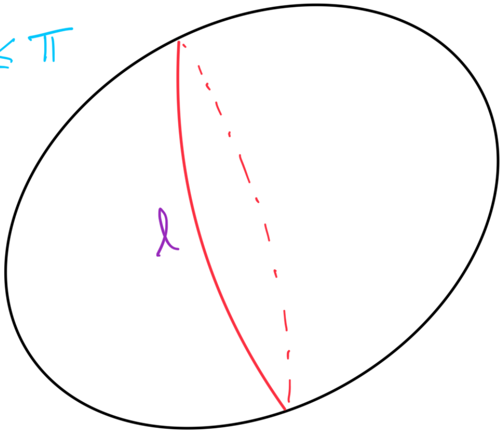
$$\frac{{\ell_{min}}^2}{Area} \leq \pi \leq \frac{{\ell_{max}}^2}{Area}$$

If equality holds in one of the inequalities then the metric is Zoll.

This answers positively a conjecture of Babenko.

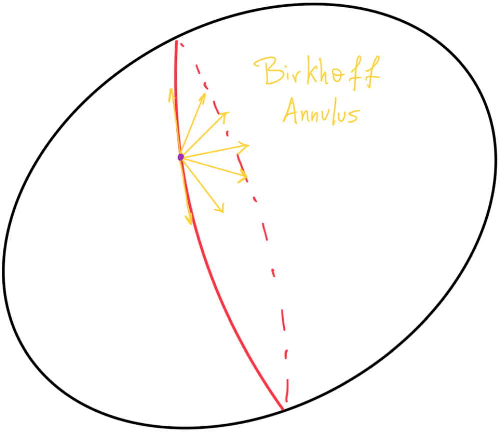
Applications I: Systolic Geometry on the two-sphere

$$\frac{\ell^2}{\text{Area}} \leq \pi$$

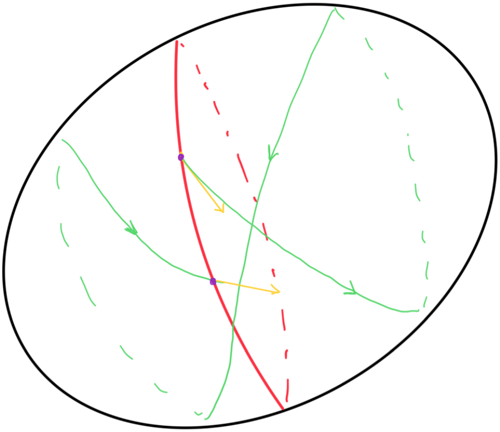


Equality $\Longleftrightarrow$ Zoll

Applications I: Systolic Geometry on the two-sphere



Applications I: Systolic Geometry on the two-sphere



Applications II: Systolic inequalities for contact forms

Theorem (Saglam, ABHS). For every closed contact manifold (M^{2n-1}, ξ) and every $C > 0$ there exists a contact form λ satisfying

$$\xi = \ker \xi \qquad \frac{{T_{min}}^n}{vol} \geq C$$

Systolic inequalities for contact forms are not phenomena in contact topology.

Theorem (ABHS). For every $\epsilon > 0$ there is a dynamically convex contact form λ on S^3 satisfying

$$(2 - \epsilon) < \frac{T_{min}(\lambda)^2}{vol(\lambda)} < 2$$

The "weak Viterbo conjecture", originally made for convex bodies, can not be extended to the dynamically convex case.

Corollary (ABHS). There exists $\Omega \subset \mathbb{R}^4$ smooth, compact, star-shaped domain, such that $\partial\Omega$ is dynamically convex, where uniqueness of capacities fails.

Proof. By the previous result, the Gromov width can be as small as (almost) $\frac{1}{\sqrt{2}}$ times any capacity given in terms of periods of closed Reeb orbits. □

Applications III: Symplectic Topology

Ω = smooth, compact domain in $\mathbb{R}^4$
$\omega_0 = \sum dq \wedge dp$ standard symplectic structure

$$\mathcal{A}_{Hopf}(\Omega, \omega_0) = \inf \left\{ \int_D \omega_0 \;\middle|\; \begin{array}{l} D \subset \partial\Omega \text{ disk-like GSS} \\ \text{for the characteristic flow} \end{array} \right\}$$

Theorem (H., Hutchings, Ramos). $\exists$ symplectic capacity defined by

$$c_{Hopf}(X, \omega) = \sup \left\{ \mathcal{A}_{Hopf}(\Omega, \omega_0) \;\middle|\; \begin{array}{l} (\Omega, \omega_0) \overset{s}{\hookrightarrow} (X, \omega) \\ \Omega \text{ star-shaped} \\ \partial\Omega \text{ dynamically convex} \end{array} \right\}$$

Applications III: Symplectic Topology

Theorem (H., Hutchings, Ramos).
$\Omega \subset (\mathbb{R}^4, \omega_0)$ smooth, compact, star-shaped
$\partial\Omega$ dynamically convex

$$\Rightarrow \quad c_{Hopf}(\Omega, \omega_0) = c_1^{ECH}(\Omega, \omega_0)$$

Remarks.

- $c_{Hopf} = c_1^{ECH}$ should be always true.
- This can be thought of as a way of understanding c_1^{ECH} purely in terms of pseudo-holomorphic curves (no SW-theory).

Applications III: Symplectic Topology

c_Z = the cylindrical capacity
Ω = smooth, compact, star-shaped domain in $\mathbb{R}^4$

Theorem (Edtmair).

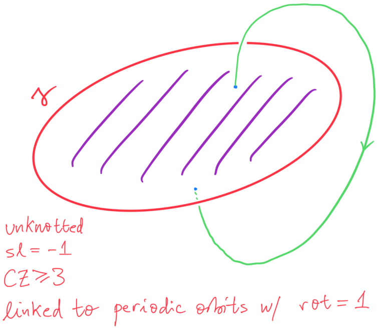
- If $\partial\Omega$ dynamically convex then

$$c_Z(\Omega,\omega_0) = c_{Hopf}(\Omega,\omega_0) = \inf \left\{ \int_D \omega_0 \;\middle|\; D \subset \partial\Omega \text{ disk-like GSS} \right\}$$

- Ω is C^3-close to the unit ball $\Rightarrow$ All symplectic capacities coincide on (Ω,ω_0).

Theorem (ABHS).
Ω is C^3-close to the unit ball $\Rightarrow$ Weak Viterbo conjecture holds.

An existence statement



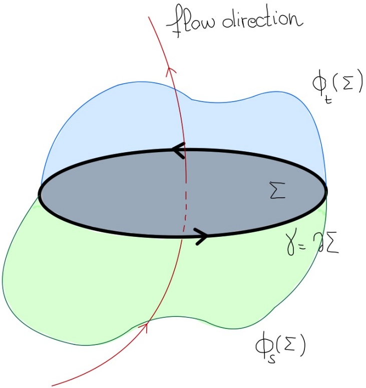
Theorem (H., Salomão). These are conditions sufficient for γ to span a disk-like GSS on (S^3, ξ_{std}). Moreover, they are C^∞-generically necessary.

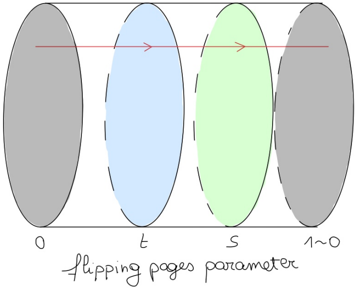
Questions.

- Examples of non-degenerate contact forms on (S^3, ξ_{std}) with infinitely many closed Reeb orbits?
- Examples of bumpy Riemannian metrics on S^2?

It is well-known that such contact forms and Riemannian metrics exist abundantly, but can we describe them explicitly?

An existence statement





(A. Florio)

Applications IV: Dynamics

Theorem (Abbondandolo, Alves, Saglam, Schlenk).
For every closed (co-orientable) contact manifold (M, ξ) and every $\epsilon > 0$, $\exists$ contact form such that

$$vol = 1 \qquad h_{top} < \epsilon$$

(Entropy collapse)

Theorem (Chaidez, Edtmair).
$\exists$ starshaped $\Omega \subset \mathbb{R}^4$, with $\partial\Omega$ dynamically convex, such that Ω is NOT symplectomorphic to a domain with strictly convex boundary.

(GSSs, Ruelle invariant and Riemannian geometry)

In higher dimensions: strong results by Ginzburg-Macarini.

Applications IV: Dynamics

Symmetric orbits

Theorem (Frauenfelder, Kang + Kang). If a dynamically convex energy level in $(\mathbb{R}^4, \omega_0)$ is conjugation symmetric, then there are two or infinitely many conjugation symmetric closed orbits.

This applies to certain regimes of the planar restricted 3-body problem.

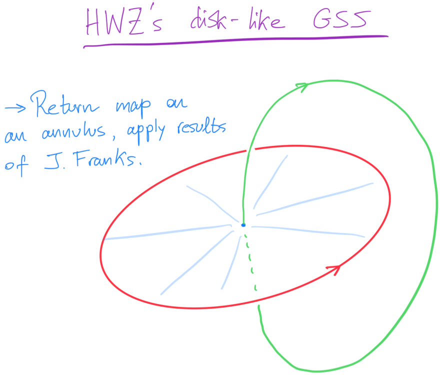
Applications IV: Dynamics

Can we actually understand the global structure of a Reeb flow on a closed 3-manifold?

Applications IV: Dynamics

Theorem (HWZ). Dynamically convex Reeb flows on S^3 have disk-like GSS.

Corollary (HWZ). Dynamically convex contact forms on S^3 have 2 or ∞-many closed Reeb orbits.



Applications IV: Dynamics

It is conjectured that "$2/\infty$" holds for every Reeb flow on a closed 3-manifold. The validity of $2/\infty$ is a strong statement, since it is a statement about all Reeb flows.

But one might argue that one does not learn much about the structure of a given Reeb flow from knowing that it has ∞-many closed orbits.

However, one does get a complete structural description in the case of exactly 2 closed orbits.

Theorem (Cristofaro-Gardiner, H., Hutchings, Liu). Reeb flows on closed 3-manifolds with exactly 2 periodic orbits can only exist on standard lens spaces (or S^3), they always admit rational disk-like GSS, with return maps given by *pseudo-rotations*.

Corollary. Periodic orbits are *irrationally elliptic*, the link formed by them can be characterized, there are precise relations between their *actions*, *rotation numbers* and the *contact volume*.

Consequence. All structural statements and conjectures about pseudo-rotations have now their versions for Reeb flows with exactly 2 orbits.

Knots and links of periodic orbits

In a celebrated paper, Birman and Williams propose the study of knots and links of periodic orbits as a way of understanding dynamics from a qualitative viewpoint.

Topology Vol. 22, No. 1, pp. 47–82, 1983
Printed in Great Britain.
0040–9383/83/010047–36$03.00/0
Pergamon Press Ltd.

KNOTTED PERIODIC ORBITS IN DYNAMICAL SYSTEMS—I: LORENZ'S EQUATIONS

JOAN S. BIRMAN† and R. F. WILLIAMS‡

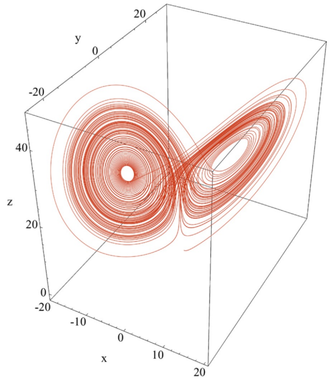
(*Received* 31 *March* 1980)

§1. INTRODUCTION

THIS PAPER is the first in a series which will study the following problem. We investigate a system of ordinary differential equations which determines a flow on the 3-sphere S^3 (or $\mathbb{R}^3$ or ultimately on other 3-manifolds), and which has one or perhaps many periodic orbits. We ask: can these orbits be knotted? What types of knots can occur? What are the implications?

Knots and links of periodic orbits

The study of knots and links as a mathematical subject was initiated by Lord Kelvin within the context of fluid dynamics.



(Araújo-Pacifico)

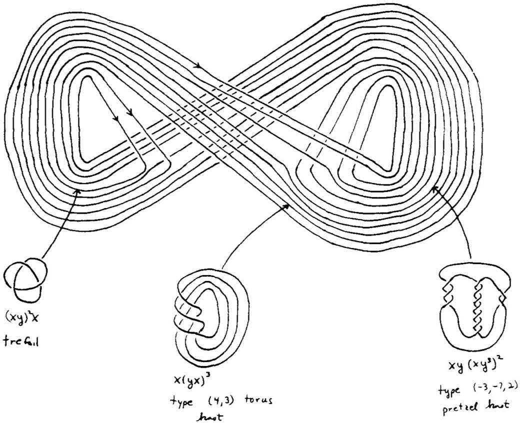
In 1963 Lorenz proposed a simplified model for atmospherical prediction.

$$\begin{cases} \dot{x} = \sigma(y - x) \\ \dot{y} = rx - y - xz \\ \dot{z} = xy - bz \end{cases}$$

(convection rolls)

Lorenz's attractor contains many periodic orbits, the Lorenz knots.

Knots and links of periodic orbits

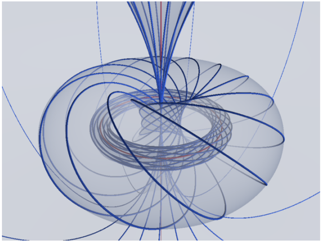


Theorem (Birman and Williams). All Lorenz knots are fibered.

Knots and links of periodic orbits

Question. In general, how bad can the knots of periodic orbits of a Reeb flow on (S^3, ξ_{std}) be?

Answer (Etnyre, Ghrist). Very! There exists a real-analytic Reeb flow on (S^3, ξ_{std}) that realizes all knot types simultaneously.



(by P. Massot)

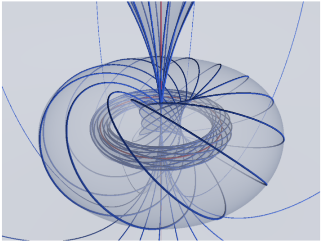
Only one knot type:
The Hopf flow!

Question. As we 'move away' from the Hopf flow, can we understand knots and links that can be realized?

Knots and links of periodic orbits

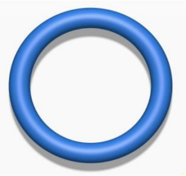
Definition (Ghys). A non-singular vector field on a homology 3-sphere is *right-handed (left-handed)* if any two Borel invariant probability measures link *positively (negatively)*.

The Hopf flow, again!



Knots and links of periodic orbits

Theorem (Ghys). Finite collections of periodic orbits of a right-handed flow span GSSs, in particular, form fibered links.



The unknot: allowed.

The figure-eight knot: allowed.

The Stevedore knot: NOT allowed.

The 10_{20} knot: NOT allowed.

Stalling's fibration theorem: Only knots γ with finitely-generated ker $link(\cdot, \gamma) \subset \pi_1(\gamma^c)$ are allowed as periodic orbits.

Knots and links of periodic orbits

Question (Ghys). What geodesic flows are left- (right-) handed ?

Only the two-sphere is allowed if we stick to closed orientable surfaces.

If orbifolds are allowed then: **Theorem (Dehornoy).** The geodesic flow on a hyperbolic 3-conic two-sphere is left-handed.

Question. What Reeb flows on (S^3, ξ_{std}) are right-handed ?

Question. Is every dynamically convex Reeb flow on S^3 right-handed ? Maybe not... but?

Knots and links of periodic orbits

$\lambda =$ dynamically convex contact form on (S^3, ξ_{std})
$\gamma =$ unknotted closed Reeb orbit, with $sl(\gamma) = -1$

Definition.

$$\kappa(\gamma) = \liminf_{T \to +\infty} \left(\inf_{x,u} \frac{\Delta\Theta(T,u)}{link(\phi^{[0,T]}(x), \gamma)} \right)$$

$\Delta\Theta(T,u) =$ angle variation along (transverse) linearized flow
$link(\phi^{[0,T]}(x), \gamma) =$ "asymptotic linking number" (for T large)

Example. For the Hopf flow, $\kappa(\gamma) = 4\pi$ for every Hopf fiber γ.

Knots and links of periodic orbits

Theorem (Florio, H.). If $\kappa(\gamma) > 2\pi$ then the Reeb flow is right-handed.

Corollary (Florio, H.). Consider the strictly convex boundary of a smooth convex body $C \subset \mathbb{R}^4$, with a disk-like GSS D.

$$K_{min}\tau_{min} > \frac{\pi}{2} \qquad \Rightarrow \qquad \text{Reeb flow is right-handed}$$

$$K_{min} = \inf_{\partial C}\{\text{sectional curvatures}\} \qquad \tau_{min} = \inf_D\{\text{return time}\}$$

Main application (Florio, H.). If a Riemannian S^2 is pinched by at least 0.7225, then the geodesic flow lifts to a right-handed dynamically convex Reeb flow on (S^3, ξ_{std}).

A big and **explicit** chunk of the set of right-handed geodesic flows of positively curved two-spheres, containing flows that can be quite far from integrable.

Genericity results for rational GSS (Birkhoff sections)

Theorem (Colin, Dehornoy, H., Rechtman).

- A C^1-generic Reeb flow on a closed 3-manifold carries a rational GSS.
- A C^∞-generic contact form on a homology 3-sphere carries a rational GSS.

Tools

- Irie's equidistribution theorem
- The action-linking lemma (Bechara-Senior, H., Salomão)
- Asymptotic cycles (Schwartzman-Fried-Sullivan theory)
- Broken book decompositions (Colin, Dehornoy, Rechtman)

Irie's equidistribution.

For a C^∞-generic contact form on a closed 3-manifold the following holds: $\exists$ sequences

$$\Gamma^n = \{\gamma_1^n, \dots, \gamma_{J(n)}^n\} \quad \text{finite sets of periodic orbits}$$

$$\{p_1^n, \dots, p_{J(n)}^n\} \subset (0,1] \quad 1 = p_1^n + \cdots + p_{J(n)}^n$$

such that

$$\sum_{j=1}^{J(n)} p_j^n \frac{(\gamma_j^n)_* Leb}{T(\gamma_j^n)} \quad \longrightarrow \quad \frac{\lambda \wedge d\lambda}{vol}$$

in the weak* topology.

Genericity results for rational GSS (Birkhoff sections)

Action-Linking Lemma (Bechara-Senior, H., Salomão).
M = homology 3-sphere
λ = contact form on M

$$\Rightarrow link\left(\frac{\lambda \wedge d\lambda}{vol}, \gamma\right) = \frac{T(\gamma)}{vol}$$

Corollary. For a C^∞-generic contact form on a homology 3-sphere M: If $h_1, \dots, h_m$ are periodic Reeb orbits, then there are periodic Reeb orbits in

$$\gamma_1, \dots, \gamma_N \subset M \setminus (h_1 \cup \cdots \cup h_m)$$

and $x_1, \dots, x_N \in \mathbb{Z}_{>0}$ such that

$$\sum_{j=1}^{N} x_j \; link(\gamma_j, h_i) > 0 \qquad \forall i$$

Genericity results for rational GSS (Birkhoff sections)

Folklore Theorem (Schwartzman-Fried-Sullivan).

$M =$ homology 3-sphere with a smooth flow
$L =$ link made of periodic orbits
$\mathcal{P}_{inv}(M \setminus L) =$ invariant Borel probability measures on $M \setminus L$
$y \in H^1(M \setminus L)$

If

$$\mu \cdot y > 0 \ \forall \mu \in \mathcal{P}_{inv}(M \setminus L) \qquad rot^y(\gamma) > 0 \ \forall \gamma \subset L$$

then

$\Rightarrow$ some sublink $L' \subset L$ spans a rational GSS.

Genericity results for rational GSS (Birkhoff sections)

Results on Transverse Foliations:

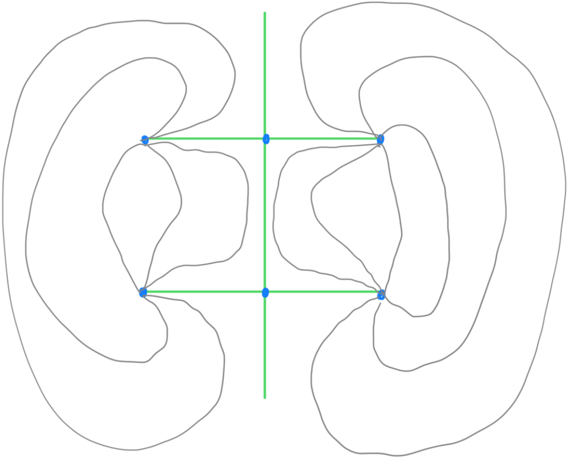
Theorem A (HWZ). A non-degenerate Reeb flow on (S^3, ξ_{std}) admits a finite-energy foliation (FEF).

Theorem B (CDR). A non-degenerate Reeb flow on a closed 3-manifold admits a broken book decomposition (BBD).

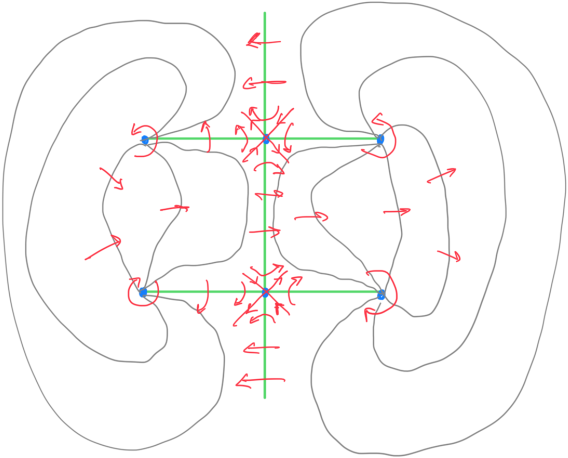
Theorem A is the pioneering result, published in the early 2000's. The curves come from Gromov-Witten theory on $(\mathbb{C}P^2, \omega_{FS})$.

In Theorem B the curves come from the relation between Seiberg-Witten theory and Hutchings' ECH.

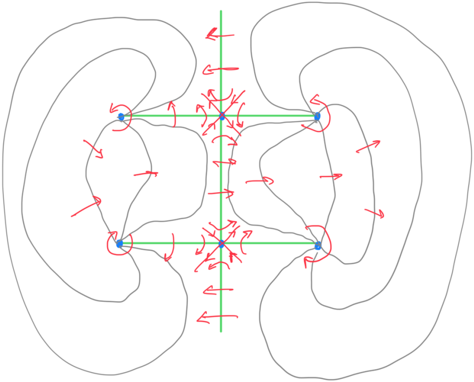
Genericity results for rational GSS (Birkhoff sections)



Genericity results for rational GSS (Birkhoff sections)



Genericity results for rational GSS (Birkhoff sections)

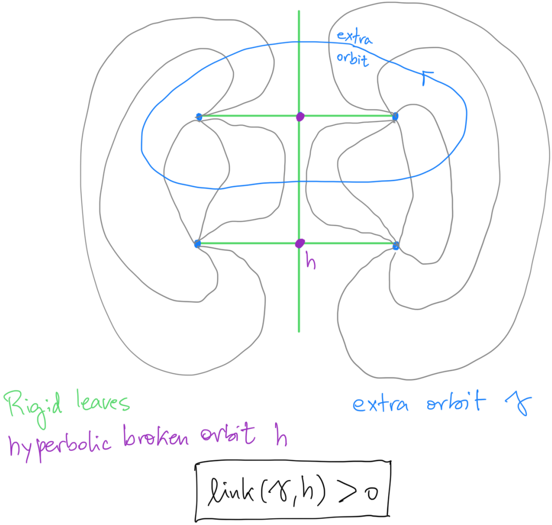


A 3-2-3 Foliation

Theorem (de Paulo, Salomão):

These exist for concrete (degenerate) Reeb flows coming from Celestial Mechanics.

Genericity results for rational GSS (Birkhoff sections)



1. Equidistribution + Action-Linking ⇒ "extra orbits"
2. Blow all orbits up (binding + extra)
3. Consider cohomology class

   $y = a$ link w/ rigid leaves

   $+$ link w/ extra orbits

   with $a \gg 1$
4. Consider ergodic measures: those in the complement of broken hyperbolic obits link positively with rigid leaves; those hidden in the linearization of the broken hyperbolic obits link positively with extra orbits;
5. Apply SFS.