Big Fiber Theorems and Ideal-Valued Measures in Symplectic Topology

Yaniv Ganor

Technion – Israel Institute of Technology, Haifa, Israel

Joint with Adi Dickstein, Leonid Polterovich and Frol Zapolsky.

Table of contents

Big Fiber Theorems

Big Fiber Theorems

Various fields of mathematics exhibit big fiber theorems:

(Template) Theorem:

For any map $f\colon X \to Y$ in a suitable class, there exists $y_0 \in Y$, such that the fiber $f^{-1}(y_0)$ is "big".

Example Theorems:

- Topological Centerpoint Theorem (Rado … Karasev);
- Maximal fiber theorem for maps of the torus (Gromov);
- Non-displaceable fiber theorem in symplectic topology (Entov-Polterovich).

Goal:

Gromov's ideal-valued measures $+$ Varolgunes' relative symplectic homology $\implies$ put all three theorems on equal footing.

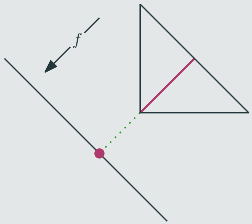
- Y - metric space of covering dimesnion d.
- p - a positive integer.

Topological centerpoint theorem, Karasev (2014)

Let $n = p(d+1)$ and let Δ^n be the n-simplex. Then for any continuous map $f\colon \Delta^n \to Y$, there exists a point $y_0 \in Y$, such that $f^{-1}(y_0)$ intersects all pd-dimensional faces of Δ^n.

For affine maps Rado (1946).

Example ($d = p = 1$, $n = 2$)



- Y - metric space of covering dimesnion d.
- p - a positive integer.

Torus Theorem, Gromov (2010)

Let $n \geq p(d+1)$. For every continuous map $f\colon \mathbb{T}^n \to Y$, there exists a point $y_0 \in Y$, such that $\operatorname{rank}\left(\check{H}^*(\mathbb{T}^n) \to \check{H}^*\left(f^{-1}(y_0)\right)\right) \geq 2^p$

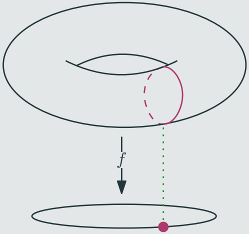
Example ($d = p = 1$, $n = 2$)

$f : S^1 \times S^1 \to S^1$, proj on the 1st factor.

$f^{-1}(y_0) = S^1$.

$\operatorname{im}\left(H^*\left(\mathbb{T}^2\right) \to H^*\left(f^{-1}(y_0)\right)\right) = \langle 1, [dy] \rangle$

$\operatorname{rank}\left(H^*\left(\mathbb{T}^2\right) \to H^*\left(f^{-1}(y_0)\right)\right) = 2 = 2^1$.



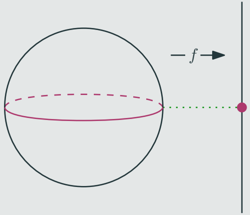
- (M^{2n}, ω) - a closed symplectic manifold.
- $\mathbf{f} = (f_1, \dots, f_N) \colon M \to \mathbb{R}^N$, such that $\{f_i, f_j\} = 0, \ \forall i, j$.

Non-displaceable Fiber Theorem, Entov-Polterovich (2006)

There exists $p \in R^N$ such that $\mathbf{f}^{-1}(p)$ is non-displaceable.

Example ($n = 1$, $N = 1$)

$\mathbf{f} = (f) \colon S^2 \to \mathbb{R}$
Height function.



Ideal Valued Measures

Ideal Valued Measures

- $(A, *)$ - a graded skew-commutative associative unital algebra.
- $\dim A < \infty$.
- X - a compact Hausdorff topological space.

Think: $A = \check{H}^*(x)$.

Definition (Ideal Valued Measure (Gromov))

An *A-ideal valued measure, (A-IVM)* is an assignment

$$U \subset X \text{ open} \mapsto \mu(U) \subset A \text{ graded ideal, such that:}$$

1. **(Normalization):** $\mu(\emptyset) = 0, \quad \mu(X) = A$.
2. **(Monotonicity):** $U \subset U' \implies \mu(U) \subset \mu(U')$.
3. **(Continuity):** If $U_1 \subset U_2 \subset \ldots$ & $U = \bigcup_i U_i$, then $\mu(U) = \bigcup_i \mu(U_i)$.
4. **(Additivity):** $\mu(U \cup U') = \mu(U) + \mu(U')$ for disjoint U, U'.
5. **(Multiplicativity):** $\mu(U) * \mu(U') \subset \mu(U \cap U')$.
6. **(Intersection):** If U, U' cover X, then $\mu(U \cap U') = \mu(U) \cap \mu(U')$.

Think: $\mu(U) = \ker\left(\check{H}^*(X) \to \check{H}^*(X \setminus U)\right)$.

Ideal Valued Measures - Examples

IVM Examples:

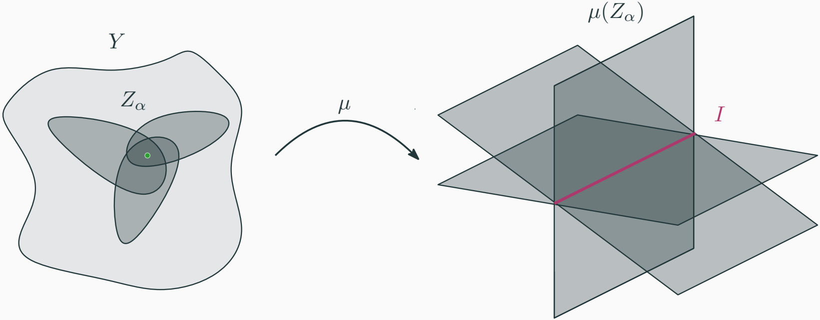
1. **Čech cohomology IVM** - $\mu(U) = \ker\left(\check{H}^*(X) \to \check{H}^*(X \setminus U)\right)$.
2. **Pushforward IVMs** - Given an IVM μ on X, and $f : X \to Y$ cont.
   Obtain an IVM on Y: $f_*\mu(U) := \mu\left(f^{-1}(U)\right)$ for all $U \subset Y$ open.

Ideal Valued Measures - Abstract Centerpoint Theorem (Karasev)

Theorem (Variation on Karasev, 2014)

Let Y be a compact metric space of covering dim. d.
$(A, *)$ - an algebra. I - an ideal s.t. $I^{*(d+1)} \neq 0$. μ - an IVM on Y. Then:

$$\bigcap \left\{ \underset{cpct}{Z} \subset Y \,\middle|\, I \subset \mu(Z) \right\} \neq \emptyset$$



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$$\bigcap \left\{ \underset{cpct}{Z} \subset Y \middle| I \subset \mu(Z) \right\} \neq \emptyset$$

Sketch.

Otherwise $(Z^c)_{I \in \mu(Z)}$ is an open cover.
By Palais lemma, and the covering dim. $\exists$ a refinement $\left(V_{ij}\right)_{ij}$, such that $i = 0, \ldots, d$, and $\forall i$, the sets $\left(V_{ij}\right)_j$ are pairwise disjoint.
Put $K_i = \cap_j V_{ij}^c$.
(Multiplicativity) $\implies \prod_{i=0}^{d} \mu(K_i) \subset \mu(\cap_{i=0}^{d} K_i) = \mu(\emptyset) = 0$.

So enough to show: $I \subset \mu(K_i)$, $\implies 0 \neq I^{d+1} \subset 0$.
Follows by (Intersection) and (Monotonicity).
Contradiction!

Corollary (Variation on Karasev, 2014)

Y a d-dim space, $I^{d+1} \neq 0$ as before.
But now, X – any compact Hausdorff space. μ - IVM on X.
Any continous map $f\colon X \to Y$ has a fiber intersecting every compact $Z \subset X$ with $I \subset \mu(Z)$.

Follows by applying the abstract centerpoint theorem to the pushforward IVM: $f_*\mu$.

- Y - metric space of covering dimesnion d.
- p - a positive integer.

Topological centerpoint theorem, Karasev (2014)

Let $n = p(d+1)$ and let Δ^n be the n-simplex. Then any continuous $f\colon \Delta^n \to Y$, has a fiber $f^{-1}(y_0)$ intersecting **all** pd-dim faces of Δ^n.

Proof (Sketch).

We need an algebra A and an IVM μ on Δ^n, s.t:

- $\exists$ ideal $I \subset A$ s.t. $I^{d+1} \neq 0$.
- For every pd-dimensional face σ, $I \subset \mu(\sigma)$.

Consider the moment map $\Phi\colon \mathbb{C}P^n \to \Delta^n$.
Preimage of face of Δ is a complex projective hyperspace of the same complex dim.

Take $\mu = \Phi_* \nu$, where is the cohomological IVM on $\mathbb{C}P^n$,
$I = \langle PD[\mathbb{C}P^{pd}]\rangle$.

□

Ideal Valued Measures - Gromov's Torus Theorem

The proof has a similar structure.

Two ingredients

- A suitable abstract centerpoint theorem:
  "There exists a point y_0 with $\operatorname{codim} \mu(Y \setminus y_0) \geq$ *[something]".*
- Pushforward IVMs.

Ideal Valued Quasi Measures

Goals:

- Adapt IVMs to the symplectic setting.
- Be able to apply centerpoint theorems.
- Explore symplectic rigidity through the ideal-valued lens.

Commuting Subsets and Involutive Maps

- (M, ω) - a closed symplectic manifold.

Definition

A map $f = (f_1, f_2, \ldots, f_k) : M \to \mathbb{R}^k,$ where $\{f_i, f_j\} = 0$ for all i, j is called *involutive*.

Definition

More generally a smooth map $f \colon M \to B$ is called *involutive* if

$\{f^* F, f^* G\} = 0$

for all $F, G \in C^\infty(B)$.

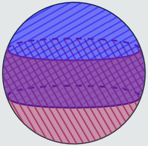
Remark: One can always embed B into $\mathbb{R}^N$ and use the first definition.

Commuting Subsets and Involutive Maps

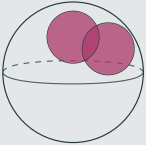
Definition

Say that compact $K, K' \subset M$ *commute* if there exist Poisson commuting $f, g \in C^\infty(M)$ with $K = f^{-1}(0)$, $K' = g^{-1}(0)$.
We say that open sets commute if their complements commute.

Example



Non intersecting boundaries.
Commuting.



Boundaries intersect.
Doesn't commute.
(True in $\dim 2$).

Ideal Valued Quasi Measures

- (M, ω) - a closed symplectic manifold.

Definition (Ideal Valued Quasi Measure)

An *A-ideal valued quasi measure, (A-IVQM)* is the same as an A-IVM, **except for multiplicativity**, which is replaced by the weaker:

1. **(Quasi-Multiplicativity):** $\tau(U) * \tau(U') \subset \tau(U \cap U')$,
   if U and U' commute.

Ideal Valued Quasi Measures

To adapt to the symplectic setting, we require two extra axioms:

2. **(Invariance):** $\tau(U) = \tau(\phi(U))$ for $\phi \in \mathsf{Symp}_0(M)$.
3. **(Vanishing):** If a compact K is (Hamiltonianly) displaceable, then there exists $U \supset K$ with $\tau(U) = 0$. Moreover $\tau(M \setminus K) = A$.

Ideal Valued Quasi Measures - Main Theorem

Theorem (Dickstein–G–Polterovich–Zapolsky)

Let (M, ω) be a closed symplectic manifold. Then there exists an A-IVQM on M, for some algebra A.

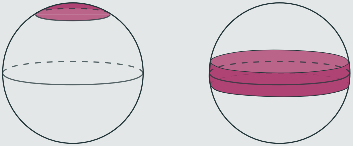
Upshot: Preimages of sets under involutive maps commute, hence:
IVQMs push to IVMs under involutive maps!

— Gain symplectic analogues to Karasev's and Gromov's theorems.

Ideal Valued Quasi Measures - Example

Example (In dim 2: take $M = S^2$ of area$= 1$)

- Enough to define IVQM on 2-dim closed connected submanifolds with boundary Q.
- $$\tau(Q) = \begin{cases} 0 & Q \text{ is contained in a smooth closed disc of area } < 1/2 \\ A & \text{else} \end{cases}$$



$\mu(Q) = 0.$ $\mu(Q) = A.$

Ideal Valued Quasi Measures - General Construction - I

Relative symplectic cohomology (Varolgunes):

- A homology $SH(K)$ for every compact $K \subset M$.
- Also, a ring. (Varolgunes-Tonkonog).
- Restriction maps $SH(K) \to SH(K')$.
- Mayer Vietoris sequence for commuting pairs.

$$SH(A \cup B) \longrightarrow SH(A) \oplus SH(B) \longrightarrow SH(A \cap B) \xrightarrow{+1} SH(A \cup B)$$

- Vanishes for displaceable sets.

Ideal Valued Quasi Measures - General Construction - II

Our IVQM

For a compact K: $\tau(K) = \ker\left(SH^*(M) \to SH^*(M \setminus U)\right)$.

Remarks:

- Can either discuss IV(Q)Ms on compacts or on open sets.
- To achieve continuity one has to alter this definition a bit.
- Quasi-multiplicativty is nontrivial and requires new ideas.

Big Fiber Theorems, Revisited

IVQMs - Non Displaceable Fiber - Analog to Gromov's Thm

A quantitative version of Entov-Polterovich non displaceable fiber:

Theorem (Dickstein–G–Polterovich–Zapolsky)

Every involutive map $f\colon M \to B$ has a fiber $f^{-1}(b_0)$ with $\operatorname{codim} \tau(M \setminus f^{-1}(b_0))$ at least 1.

- Displaceabliliy implies $\operatorname{codim} \tau \left(M \setminus f^{-1}(b_0)\right) = 0$.
- Gromov gives lower bounds for $\operatorname{codim} \tau(M \setminus f^{-1}(b_0))$.

IVQMs - Symplectic Centerpoint - Analog to Karasev's Thm

Given:

- I - graded ideal, $I^{d+1} \neq 0$.
- B of covering dim d.

Theorem (Dickstein–G–Polterovich–Zapolsky)

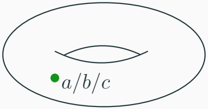
Every involutive map $f : M \to B$ has a fiber intersecting all members of the collection:

$$\left\{ \underset{cpct}{Z} \subset M \,\middle|\, I \subset \tau(Z) \right\}$$

— A source for a new kind of examples of symplectic rigidity.

IVQMs - Symplectic Centerpoint - Concrete Example

Take the torus $\mathbb{T}^6$, with coordinates $p_i, q_i \in \mathbb{T}^2$, $\omega = \sum dp \wedge dq$.
For every $a, b, c \in \mathbb{T}^2$ consider the following coisotropic subtori in $\mathbb{T}^6$:



$$T_1(a) = \{(\mathbf{p}, \mathbf{q}) \mid (q_1, q_2) = a\},$$
$$T_2(b) = \{(\mathbf{p}, \mathbf{q}) \mid (p_1, p_3) = b\},$$
$$T_3(c) = \{(\mathbf{p}, \mathbf{q}) \mid (p_2, q_3) = c\}.$$

Set $T(a, b, c) = T_1(a) \cup T_2(b) \cup T_3(c)$.

Theorem (Dickstein–G–Polterovich–Zapolsky)

Every involutive map $\mathbb{T}^6 \times S^2 \to Y^2$ has a fiber intersecting all sets of the form:

$$T(a, b, c) \times \textit{equator}.$$

An *equator* in S^2 is any loop dividing S^2 to two discs of equal area.

The involutivity is essential:

Project $\pi\colon \mathbb{T}^6 \times S^2 \to S^2$.
For $y_0 \in S^2$ and $L \subset S^2$ an equator not containing y_0,
the fiber $f^{-1}(y_0)$ disjoint from any $T(a, b, c) \times L$.

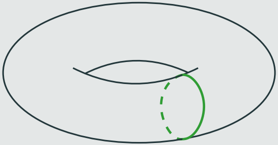
Rigidity

- (M, ω) - a closed symplectic manifold.
- τ - The $SH(M)$-IVQM on M.

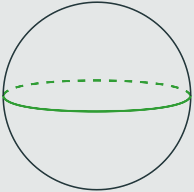
Definition

A compact $K \in M$ is *SH-heavy* if $\tau(K) \neq 0$.

Example



$\tau(K) =$ same as in cohomological IVM

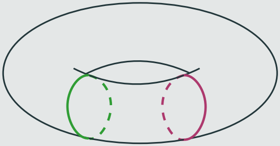


$\tau(K) =$ full measure

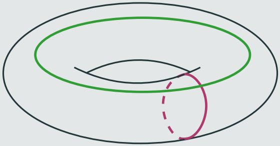
Properties

- SH-heavy sets are Ham non-displaceable.
- For K, K', if $\tau(K) * \tau(K') \neq 0$ then:
  — K, and K' are SH-heavy.
  — K is Symp_0 non-displaceable from K'.

Example



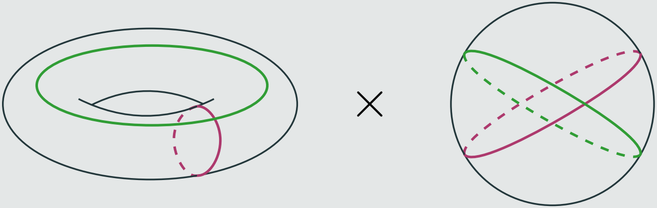
$\tau(K) * \tau(K') = 0$



$\tau(K) * \tau(K') \neq 0$

SH-Heaviness - Nontrivial Symplectic Example

Example



Two product tori in $\mathbb{T}^2 \times S^2$, in green and cherry,
Non displaceable from each other.

Note: They are smoothly displaceable.

SH-Heaviness - Proof

$\tau(K) * \tau(K') \neq 0 \implies K$ **is non-displaceable from** K':

Proof.

Assume $\phi \in \mathsf{Symp}_0$ displaces K from K': $\quad \phi(K) \cap K' = \emptyset$.
Then $\phi(K)$ and K commute.
(Quasi-Multiplicativity) $\implies$

$$\tau\big(\phi(K)\big) * \tau(K') \subset \tau\big(\phi(K) \cap K'\big) = \tau(\emptyset) = 0.$$

(Invariance) $\implies$ $\tau(\phi(K)) = \tau(K)$, hence
$0 \neq \tau(K) * \tau(K') \subset 0.$ Contradiciton! □

Categorification of Heaviness?

- (M,ω) - a closed symplectic manifold.
- e - an idempotent in $QH(M)$.

Definition (Heavy sets (Entov and Polterovich))

$F \in C^\infty(M) \mapsto \zeta(F) := \lim_{k\to\infty} \frac{c(kF;e)}{k}$

Partial symplectic quasi state

$K \subset M$ is *heavy* if $\forall F \in C^\infty(M)$, one has $\zeta(F) \geq \inf_K F$.

Properties

- Heavy sets are non-displaceable.
- Heavy sets need not necessarily intersect! (e.g. two parallel meridians on $\mathbb{T}^2$).
- Unclear how to detect intersections, in contrast to SH-heavy sets.

Conjecture: Heavy $\implies$ SH-heavy.

Proven for a simple case:

index bounded incompressible domains in aspherical manifolds.

The other direction is more speculative.

Construction of IVQMs

The Novikov field — The Novkiov ring

$$\Lambda = \left\{ \sum_{i=0}^{\infty} c_i T^{\alpha_i} \,\middle|\, c_i \in \mathbb{Q},\ \alpha_i \in \mathbb{R},\ \alpha_i \nearrow \infty \right\}, \quad \Lambda_{\geq 0} = \left\{ \sum_{i=0}^{\infty} c_i T^{\alpha_i} \in \Lambda \,\middle|\, \alpha_i \geq 0 \right\}$$

- H - a Hamiltonian,
- $\mathcal{P}(H)$ - 1-periodic orbits. (graded by mod-2 CZ-index)

Floer Complex: $CF(H) := \bigoplus_{\gamma \in \mathcal{P}(H)} \Lambda_{\geq 0} \cdot \gamma.$

— Note: no cappings.

Floer Differential:

— Positive gradient flow of action functional (cohomology).

— Weighted by the topological energy of Floer solutions:

$$d\gamma_- = \sum_{\gamma_+ \in \mathcal{P}(H)} \sum_{B \in \pi_2(M, \gamma_-, \gamma_+)} \#\mathcal{M}(\gamma_-, \gamma_+) T^{\omega(B) + \int_{\gamma_+} H - \int_{\gamma_-} H} \gamma_+$$

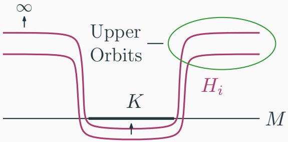
↑ "action difference"

Continuation Maps:

— Also weighted by top. energy ("action difference").

— Defined over $\Lambda_{\geq 0}$ for $H_1 \leq H_2$ (going from low to high).

- $K \subset M$ compact.
- H_i Hamiltonians adapted to K:



Symplectic Cohomological Complex: $SC(K) := \widehat{\varinjlim_{i \to \infty} CF(H_i)}$

— Completion: $\widehat{A} := \varprojlim_{r \to \infty} \left(A \otimes_{\Lambda_{\geq 0}} {}^{\Lambda_{\geq 0}}\!/\!_{\Lambda_{\geq r}} \right)$.

— Eliminates contributions of upper orbits, since, e.g.:

$$\begin{array}{ccccccccc} \Lambda_{\geq 0} & \xrightarrow{T\cdot} & \Lambda_{\geq 0} & \xrightarrow{T\cdot} & \Lambda_{\geq 0} & \xrightarrow{T\cdot} & \dots & \longrightarrow & \varinjlim \\ \wr\!\!\mid & & \wr\!\!\mid & & \wr\!\!\mid & & & & \wr\!\!\mid \\ \Lambda_{\geq 0} & \hookrightarrow & \Lambda_{\geq -1} & \hookrightarrow & \Lambda_{\geq -2} & \hookrightarrow & \dots & \longrightarrow & \Lambda \end{array}$$

Symplectic Cohomology:

$SH(K; \Lambda_{\geq 0}) := H^*(SC(K))$, – a $\Lambda_{\geq 0}$-Module.

$\boldsymbol{SH(K) := SH(K; \Lambda_{\geq 0}) \otimes_{\Lambda_{\geq 0}} \Lambda}$. – (eliminates torsion. "finite bars")

Relative Symplectic Cohomology - Properties

- **Restriction maps:** For $K' \subset K \subset M$ compacts, there are restriction maps $SH(K) \to SH(K')$.

- **Mayer-Vietoris (Varolgunes):** For $A, B \subset M$ compact commuting subsets, there exists an exact triangle:

$$\begin{array}{ccc} SH(A \cup B) & \longrightarrow & SH(A) \oplus SH(B) \\ {}_{+1}\nwarrow & & \swarrow \\ & SH(A \cap B) & \end{array}$$

- **Product (Tonkonog-Varolgunes):** $SH(K) \otimes SH(K) \xrightarrow{*} SH(K)$, compatible with restriction, making $SH(K)$ a unital ring. Moreover, $SH(M) = QH(M)$.

Ideal Valued Quasi Measures - Construction

Recall our IVQM:

- For a compact K: $\tau(K) = \bigcap_{\substack{U \supset K \\ \text{open}}} \ker\left(SH^*(M) \to SH^*(M \setminus U)\right).$

Quasi-multiplicalivity is nontrivial:

We define SH of a pair $K' \subset K$:
$SH(K, K') = H^*\left(\text{cocone}\left(SH(K) \to SH(K')\right)\right)$

$$\text{cocone}(r\colon V \to W) := \left(V \oplus W[-1], \begin{pmatrix} d_V & 0 \\ r & d_W \end{pmatrix}\right).$$

cocone = "homotopy kernel".

We have an exact triangle: $SH(K, K') \to SH(K) \to SH(K') \xrightarrow{+1}$

Main ingredient: lift the product to pairs, for A, B commuting:

$$\begin{array}{ccc} SH(M, A) \otimes SH(M, B) & & SH(M, A \cup B) \\ \downarrow & & \downarrow \\ SH(M) \otimes SH(M) & \xrightarrow{\quad * \quad} & SH(M) \end{array}$$

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SH of Pairs - Why The Sets Have to Commute in the Product?

Varolgunes' Mayer-Vietoris: $SH(A \cup B) \to SH(A) \oplus SH(B) \to SH(A \cap B) \xrightarrow{+1}$
Requires: A, B commuting.

Algebraic Topology: A, B satisfy M-V, $\iff$ (A, B) is an excisive-pair:
The natural chain map $C_*(A) + C_*(B) \to C_*(A \cup B)$ is an isomorphism in homology.

"commuting" is the symplectic analogue of "excisive-pair"

Similarly, Classically **relative cup product** exists for an excisive-pair:

$$H^*(M, A) \otimes H^*(M, B) \to H^*(M, A \cup B)$$

Expect: A, B should commute for:

$$SH(M, A) \otimes SH(M, B) \to SH(M, A \cup B)$$

Thank You!

Questions?

Ideal Valued Quasi Measures - General Construction - II

We define our IVQM τ by:

- For a compact K: $\tau(K) = \bigcap_{\substack{U \supset K \\ \text{open}}} \ker\left(SH^*(M) \to SH^*(M \setminus U)\right)$,
- For an open U: $\tau(U) = \bigcup_{\substack{K \subset U \\ \text{compact}}} \tau(K)$.

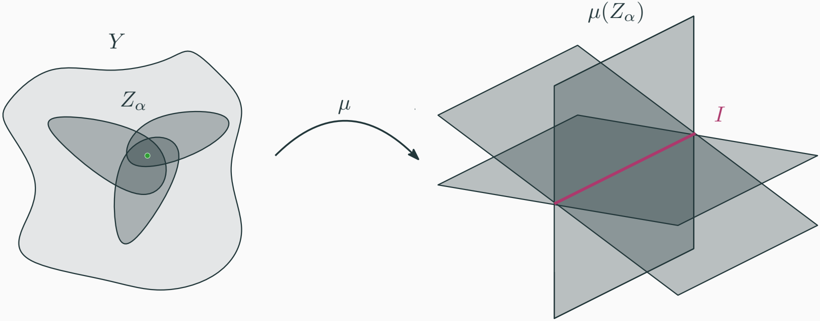
Remark: quasi-multiplicativty is nontrivial and requires new ideas.

Ideal Valued Measures - Abstract Centerpoint Theorem (Karasev)

Theorem ([Variation on Karasev, 2014)

Let Y be a compact metric space of covering dim. d.
$(A, *)$ - an algebra. I - an ideal s.t. $I^{*(d+1)} \neq 0$. μ - an IVM on Y. Then:

$$\bigcap \left\{ \underset{cpct}{Z} \subset Y \,\middle|\, I \subset \mu(Z) \right\} \neq \emptyset$$



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Proof.

Otherwise $(Z^c)_{I \in \mu(Z)}$ is an open cover.
By Palais lemma, and the covering dim. $\exists$ a refinement $\left(V_{ij}\right)_{ij}$, such that $i = 0, \ldots, d$, and $\forall i$, the sets $\left(V_{ij}\right)_j$ are pairwise disjoint.
(Monotonicity) $\implies \mu(V_{ij}^c) \supset \mu(Z_{ij}) \supset I$.
Note that $V_{ij}^c \cup V_{ij'}^c = Y$, for $j \neq j'$, and put $K_i = \cap_j V_{ij}^c$.
(Intersection) $\implies \mu(K_i) = \mu(\cap_j V_{ij}^c) = \cap_j \mu(V_{ij}^c) \supset I$.
(Product) $\implies 0 \neq I^{d+1} \subset \prod_{i=0}^{d} \mu(K_i) \subset \mu(\cap_{i=0}^{d} K_i) = \mu(\emptyset) = 0$.
Contradiction! $\square$

Ideal Valued Measures - Topological Centerpoint Theorem - I

Corollary (Variation on Karasev, 2014)

Let X be a compact Hausdorff space.
$(A, *)$ - an algebra. I - an ideal s.t. $I^{*(d+1)} \neq 0$. μ - an IVM on X.
Let Y be a compact metric space of covering dim. d.
Then any continous map $f: X \to Y$ has a fiber intersecting every compact $Z \subset X$ with $I \subset \mu(Z)$.

Proof.

Consider the pushforward IVM, $f_*\mu$, on Y, defined by
$f_*\mu(U) := \mu(f^{-1}(U))$ for $U \subset Y$ open.
If $Z \subset X$ is such that $I \subset \mu(Z)$ then $I \subset f_*\mu(f(Z))$, since:

$$I \subset \mu(Z) \subset \mu(f^{-1}(f(Z))) = f_*\mu(f(Z)).$$

By the abstract centerpoint theorem for $f_*\mu$, there exists

$$y_0 \in \bigcap_{I \subset f_*\mu(W)} W \neq \emptyset.$$

In particular, $\forall$ such Z, $y_0 \in f(Z)$, namely $f^{-1}(y_0) \cap Z \neq \emptyset$, as claimed. □

Ideal Valued Measures - Gromov's Torus Theorem

The proof has a similar structure.

Two ingredients

- A suitable abstract centerpoint theorem:
  "There exists a point y_0 *with* $\operatorname{codim} \mu(Y \setminus y_0) \geq$ *[something]".*
- Pushforward IVMs.