The smooth closing lemma for area-preserving surface diffeomorphisms

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Pugh's closing lemma in higher regularity

Fix a smooth closed, oriented surface Σ with area form ω_Σ of area 1. Let $\mathrm{Diff}(\Sigma, \omega_\Sigma)$ be the group of area-preserving diffeomorphisms.

Question (Poincaré (?), Smale)

Does a generic element of Diff(Σ, ω_Σ) have a dense set of periodic points?

- Yes in the C^0 topology (exercise).
- Pugh–Robinson (60s-80s): Yes in the C^1 topology.
- Smale's Problem #10: Are there results like these in higher regularity?

The smooth closing lemma

The answer is yes for area-preserving smooth diffeomorphisms of closed surfaces.

Theorem (Cristofaro-Gardiner–P.–Zhang '21)

Fix any $\phi \in Diff(\Sigma, \omega_\Sigma)$. Then for any open $U \subset \Sigma$ and C^∞ neighborhood $V \subset Diff(\Sigma, \omega_\Sigma)$, there is $\phi' \in V$ such that ϕ' has a periodic orbit in U.

The generic density theorem follows from the above by a Baire category argument.

Other work

- Asaoka–Irie '15 established the theorem for $\Sigma = S^2$ (and for Hamiltonian diffeomorphisms in higher genus).
- Edtmair–Hutchings '21 established the theorem simultaneously for $\Sigma = T^2$ using different, but related methods. Also established the theorem assuming the PFH-theoretic "U-cycle condition".
- Cristofaro-Gardiner–Pomerleano–P.–Zhang '21 later showed that the U-cycle condition holds in higher genus; Edtmair–Hutchings' proof extends to an alternate proof of the closing lemma.

The main actors

Our proof synthesizes three bodies of work:

- Periodic Floer homology (PFH) and its quantitative theory (Hutchings, Cristofaro-Gardiner–Humilière–Seyfaddini)
- Seiberg-Witten-Floer homology (SWF), as constructed by Kronheimer–Mrowka.
- The isomorphism between SWF and PFH (Lee–Taubes)

Setup of twisted PFH

Fix $\phi \in \text{Diff}(\Sigma, \omega_\Sigma)$. Define the **mapping torus**

$$M_\phi = [0,1] \times \Sigma / (1,p) \sim (0, \phi(p)).$$

Has a canonical vector field $R = \partial_t$, closed two-form ω_ϕ, closed one-form dt such that $dt \wedge \omega > 0$, and plane field $V = \ker(dt)$. Fix a **positive monotone**[1] class $\Gamma \in H_1(M_\phi; \mathbb{Z})$ with **degree** $d(\Gamma) = \langle dt, \Gamma \rangle \gg 1$. Fix a **trivialized reference cycle** Θ_{ref} representing Γ.

[1]Exists for arbitrarily large degree when ω is a rational multiple of a real class.

Definition of twisted PFH

The **twisted PFH** $\mathrm{TwPFH}_*(\phi, \Gamma, \Theta_{\mathrm{ref}})$ is the homology of a chain complex

$$\mathrm{TwPFC}_*(\phi, J, \Gamma, \Theta_{\mathrm{ref}}).$$

Generators are pairs (Θ, W). $\Theta = \{(\gamma_i, m_i)\}$ is a set of **orbits with multiplicity** of R such that $\sum_i m_i[\gamma_i] = \Gamma$. Hyperbolic orbits are required to have $m_i = 1$. W is a class in $H_2(M_\phi, \Theta_{\mathrm{ref}}, \Theta; \mathbb{Z})$, i.e. an **equivalence class of** 2-**chains with boundary** $\Theta - \Theta_{\mathrm{ref}}$. The differential counts "ECH index 1" J-holomorphic curves in $\mathbb{R} \times M_\phi$.

Quantitative PFH

Grading: $\mathrm{TwPFH}_*(\phi, \Gamma, \Theta_{\mathrm{ref}})$ admits a $\mathbb{Z}$-grading $I(-)$ by the "ECH index relative to Θ_{ref}".

Filtration (Hutchings): $\mathrm{TwPFC}_*(\phi, J, \Gamma, \Theta_{\mathrm{ref}})$ admits a filtration by the "PFH action functional"

$$\mathbf{A}(\Theta, W) = \int_W \omega_\phi.$$

$\Rightarrow$ for any class σ in $\mathrm{TwPFH}_*(\phi, \Gamma, \Theta_{\mathrm{ref}})$ there is a "PFH spectral invariant" $c_\sigma(\phi, \Gamma, \Theta_{\mathrm{ref}})$.

A kind of Weyl law

Fix a Hamiltonian $H \in C^\infty(\mathbb{R}/\mathbb{Z} \times \Sigma)$ and $\phi \in \mathrm{Diff}(\Sigma, \omega_\Sigma)$. Set $\phi' = \phi_H^1 \circ \phi$. There is a natural diffeomorphism $M_H : M_\phi \to M_{\phi'}$. Fix classes Γ_m in $H_1(M_\phi; \mathbb{Z})$ with $d_m = d(\Gamma_m) \to \infty$ and cycles Θ_m. Fix $\Gamma'_m = (M_H)_*(\Gamma_m)$ and $\Theta'_m = M_H(\Theta_m)$.

Theorem (Cristofaro-Gardiner–P.–Zhang)

Fix any two sequences of nonzero classes $\sigma_m \in TwPFH_*(\phi, \Gamma_m, \Theta_m)$ and $\sigma'_m \in TwPFH_*(\phi', \Gamma'_m, \Theta'_m)$. Then

$$\lim_{m\to\infty} \left(\frac{c_{\sigma'_m}(\phi', \Gamma'_m, \Theta'_m) - c_{\sigma_m}(\phi, \Gamma_m, \Theta_m) + \int_{\Theta_m} Hdt}{d_m} - \frac{I(\sigma_m) - I(\sigma'_m)}{2d_m(d_m - 1 + Genus(\Sigma))} \right) = \int_{M_\phi} Hdt \wedge \omega_\Sigma.$$

Closing lemma given the Weyl law

Adapts an idea of Irie. Fix an open set U in Σ.

1. ϕ admits positive monotone classes densely in $\mathrm{Diff}(\Sigma, \omega_\Sigma)$; can assume ϕ is monotone.
2. We prove a general non-vanishing result for twisted PFH $\Rightarrow$ there are enough nonzero classes for the Weyl law to hold.

Closing lemma given the Weyl law

3. Deform ϕ by a one-parameter family $\phi^s = \phi^1_{H^s} \circ \phi$ such that $\mathrm{supp}(H^s) \subset U$ and the Weyl law RHS is positive for every $s \in [0,1]$. Suppose for the sake of contradiction that ϕ^s has no orbits in U for any s.
4. We prove a "Hofer continuity" result for the PFH invariants (due to Cristofaro-Gardiner–Humilière–Seyfaddini for $\Sigma = S^2$) which shows the PFH spectral invariants do not change. This would force the RHS of the Weyl law to be zero, contradiction.

Proof idea of the Weyl law

Theorem (Lee–Taubes '12)

Twisted PFH is isomorphic to a version of Kronheimer–Mrowka's SWF (with non-exact perturbation)

- Quantitative information in SWF can be found in the **Chern-Simons-Dirac functional**, which induces corresponding "SWF spectral invariants". These are easily computable for "reducible" classes.
- We carefully establish a relationship between the SWF and PFH spectral invariants. Key tools include **spectral flow estimates** of Taubes and Cristofaro-Gardiner–Savale, other estimates both old and new.

Bonus: other potential applications of the Weyl law

- (Cristofaro-Gardiner–Humilière–Mak–Seyfaddini–Smith '21) The Calabi homomorphism extends to the group of Hamiltonian homeomorphisms on any closed surface.
- (expected, proved by CGHMSS '21 as well) The kernel of Fathi's mass-flow homomorphism on any closed surface is simple.

Thanks for listening!