Hofer geometry of coadjoint orbits and Peterson's theorem

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Seidel representations

Let (X, ω) be a compact symplectic manifold.

Denote by $Ham(X, \omega)$ the Hamiltonian group of (X, ω).

Seidel (1997) constructed a group homomorphism

$$\Phi^X_{Seidel} : \pi_0(\Omega Ham(X, \omega)) \to (QH^*(X))^\times$$

where

- the group structure on $\pi_0(\Omega Ham(X, \omega))$ is given by pointwise multiplication in $Ham(X, \omega)$,
- $(QH^*(X))^\times$ is the multiplicative subgroup of invertible elements of $QH^*(X)$.

The construction

$f \in \Omega Ham(X, \omega) \rightsquigarrow$

$$\begin{array}{ccc} P_f(X) & := & \mathbb{C} \times X \cup \mathbb{C} \times X \Big/ (z, x) \sim (z^{-1}, f(\frac{z}{|z|}) \cdot x) \\ \downarrow & & \downarrow \\ \mathbb{C}P^1 & := & \mathbb{C} \cup \mathbb{C} \Big/ z \sim z^{-1} \end{array}$$

Known: $P_f(X)$ is a Hamiltonian fibration over $\mathbb{C}P^1$ with fibers (X, ω).

Definition

$$\Phi^X_{Seidel}([f]) := \sum_i \# \left\{ \text{holo. section in } P_f(X) \quad \mathrm{PD}(e_i) \right\} e^i q^{\text{cont.by.holo.sect.}}$$

where $\{e_i\}, \{e^i\}$ are dual bases of $H^*(X)$.

A gluing argument $\Longrightarrow \Phi^X_{Seidel}$ is a group homomorphism.

A parametrized version

Savelyev (2008) defined a ring map extending Seidel's map

$$\Phi^X_{Savelyev} : H_{-*}(\Omega Ham(X,\omega)) \to QH^*(X)$$

$f : \Gamma \to \Omega Ham(X,\omega) \rightsquigarrow$

$$\begin{array}{ccc} P_f(X) & := & \mathbb{C}\times\Gamma\times X \cup \mathbb{C}\times\Gamma\times X \Big/ (z,\gamma,x) \sim (z^{-1},\gamma, f_\gamma(\tfrac{z}{|z|})\cdot x) \\ \downarrow & & \downarrow \\ \mathbb{C}P^1 \times \Gamma & := & \mathbb{C}\times\Gamma \cup \mathbb{C}\times\Gamma \Big/ (z,\gamma) \sim (z^{-1},\gamma) \end{array}$$

$P_f(X)$ can be considered as a smooth family $\{P_{f_\gamma}(X)\}_{\gamma\in\Gamma}$ of Hamiltonian fibrations parametrized by Γ.

Definition

$$\Phi^X_{Savelyev}([f]) := \sum_i \# \left\{ \left(\gamma, \ \text{holo. section in } P_{f_\gamma}(X) \text{ through } \mathrm{PD}(e_i) \right) \right\} e^i q^{\text{cont.by.holo.sect.}}$$

A gluing argument $\Longrightarrow \Phi^X_{Savelyev}$ is a ring homomorphism.

Main theorem

Let G be a compact Lie group and $\mathcal{O}$ a coadjoint orbit of G.

Example

$G := SU(n)$. All coadjoint orbits of G are of the form

$$F\ell(k_1, \ldots, k_r; n) := \{0 \subset V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n | \dim V_i = k_i\}$$

where $0 < k_1 < \cdots < k_r < n$ are integers.

Define $\Phi^{\Omega G, \mathcal{O}}_{Savelyev}$ to be the composition

$$H_{-*}(\Omega G) \xrightarrow[\text{map}]{\text{natural}} H_{-*}(\Omega Ham(\mathcal{O})) \xrightarrow{\Phi^{\mathcal{O}}_{Savelyev}} QH^*(\mathcal{O}).$$

Theorem (C.)

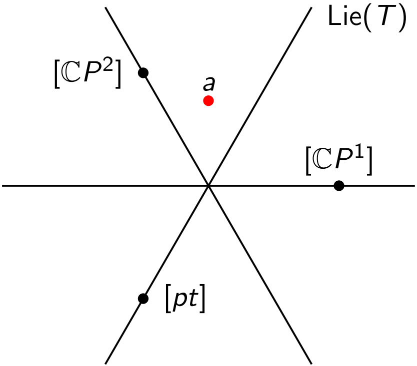
Know $\Phi^{\Omega G, \mathcal{O}}_{Savelyev}$ completely.

Basis of $H^*(\mathcal{O})$: Schubert classes

Represented by **Schubert cells** = descending manifolds wrt the Morse function $f(X) := \langle X, a \rangle$ on $\mathcal{O}$ where $a \in \mathrm{Lie}(G)^\vee$ is an generic element.

Example

$G := SU(3), \mathcal{O} := \mathbb{C}P^2$
Take a maximal torus T e.g. $T := \{\text{diagonal matrices}\}$.
Assume $a \in \mathrm{Lie}(T)$ generic. Then $\mathrm{Crit}(f) = \mathrm{Lie}(T) \cap \mathcal{O} = \{\bullet\}$.

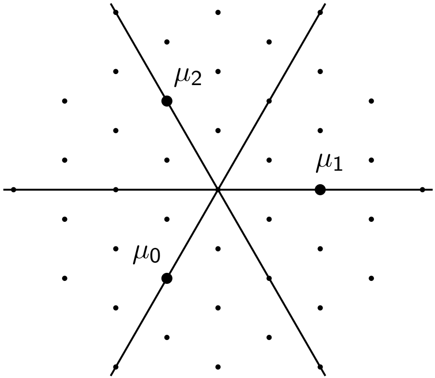


Basis of $H_*(\Omega G)$: Affine Schubert classes

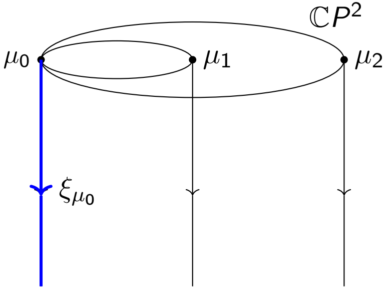
Represented by **affine Schubert cells** = descending manifolds wrt a perturbation of the energy functional E on ΩG.
Notice E is Bott-Morse whose critical set = a countable disjoint union of coadjoint orbits of G.

Example

$G := SU(3)$. Define $\{\bullet\} := \exp^{-1}(e) \cap \mathrm{Lie}(T)$, the unit lattice of $\mathrm{Lie}(T)$.



Every $\mu \in \{\bullet\}$ gives rise to an affine Schubert class ξ_μ.



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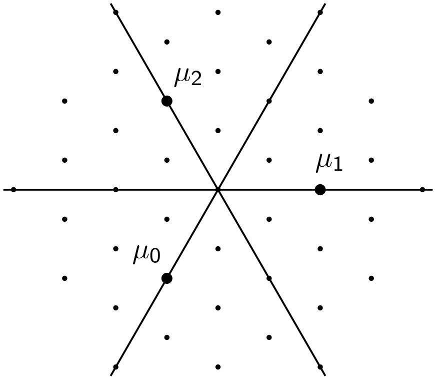

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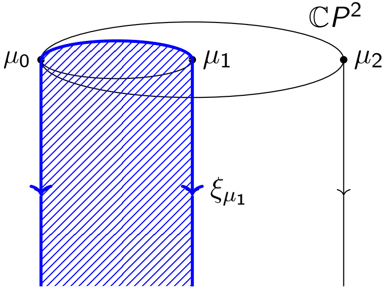

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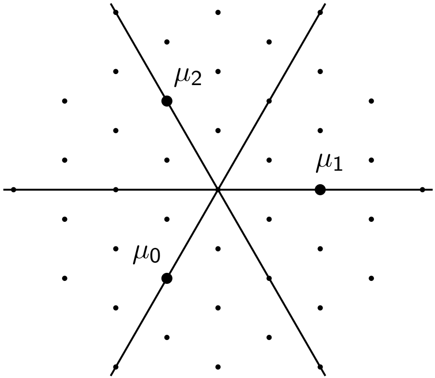

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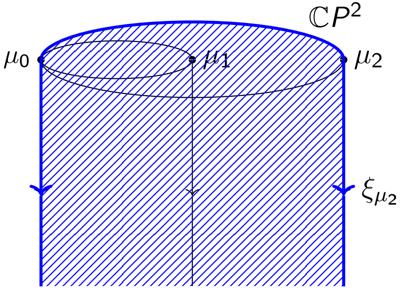


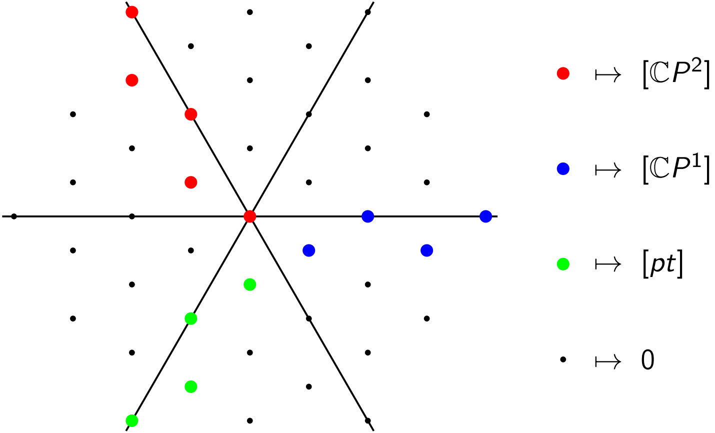
$\Phi^{\Omega G, \mathcal{O}}_{Savelyev} = ?$

Roughly, it sends a basis element to zero or an explicit basis element.

Example

$G := SU(3), \mathcal{O} := \mathbb{C}P^2$

For simplicity, set $q = 1$ ($q =$ quantum variable).



Consequence 1: Non-triviality of $\pi_*(Ham(\mathcal{O})) \otimes \mathbb{Q}$

Theorem

The dimension of the kernel of the induced map

$$\pi_*(G) \otimes \mathbb{Q} \to \pi_*(Ham(\mathcal{O})) \otimes \mathbb{Q}$$

is at most the number of facets of a Weyl chamber passing through a nearby point $\in \mathrm{Lie}(T) \cap \mathcal{O}$.

Example

This upper bound is equal to $n - 1 - r$ for the case $\mathcal{O} = F\ell(k_1, \ldots, k_r; n)$. Notice $\dim \pi_*(SU(n)) \otimes \mathbb{Q} = n - 1$.

Corollary

The induced map is injective if $\mathcal{O}$ is generic, e.g. $F\ell(1, 2, \ldots, n-1; n)$.

Remark

For generic $\mathcal{O}$, Kędra proved a much stronger result based on the work of Reznikov, Kędra-McDuff, Gal-Kędra-Tralle:

$$H^*(BHomeo(\mathcal{O}); \mathbb{Q}) \to H^*(BG; \mathbb{Q}) \text{ is surjective.}$$

Consequence 2: Hofer geometry of $Ham(\mathcal{O})$

Let (X, ω) be a compact symplectic manifold.
Let $\{\varphi_t\}$ be a path in $Ham(X, \omega)$.
There exists a unique family $\{H_t : X \to \mathbb{R}\}$, called the **normalized generating Hamiltonian** of $\{\varphi_t\}$, satisfying

$$\begin{cases} \dot{\varphi}_t &= X_{H_t} \circ \varphi_t \\ \int_X H_t \omega^{\frac{1}{2}\dim X} &= 0 \end{cases}$$

Define the **positive Hofer length functional** L^+ on $\Omega Ham(X, \omega)$

$$L^+(\{\varphi_t\}) := \int_0^1 \max_X H_t \; dt.$$

A variational problem

Given a homology class $A \in H_*(\Omega Ham(X, \omega))$, minimize

$$\max_\Gamma L^+ \circ f$$

over all smooth cycles $f : \Gamma \to \Omega Ham(X, \omega)$ representing A.

Consequence 2: Hofer geometry of $Ham(\mathcal{O})$ (cont.)

Theorem

For any $\mu \in \{\bullet, \bullet, \bullet\}$, there exists a constant C_μ such that for any smooth cycle $f : \Gamma \to \Omega Ham(\mathcal{O})$ representing ξ_μ,

$$\max_{\Gamma} L^+ \circ f \geqslant C_\mu.$$

Moreover, C_μ is attained by an explicit cycle.

The proof uses the computation of $\Phi^{\Omega G, \mathcal{O}}_{Savelyev}$ and a standard argument, e.g. Akveld-Salamon/ McDuff-Slimowitz/ McDuff-Tolman/ Savelyev etc.

Remark

Savelyev computed $\Phi^{\Omega G, \mathcal{O}}_{Savelyev}$ up to higher action terms when $\mathcal{O}$ is generic and $\mu \in$ the interior of the anti-dominant chamber. This suffices to prove the above theorem for this case.

Consequence 3: A new proof of Peterson's theorem

Theorem (Peterson 1997 unpublished, Lam-Shimozono 2010)

An explicit linear map

$$\Phi_{Peterson} : H_{-*}(\Omega G) \to QH^*(\mathcal{O})$$

is a ring homomorphism.

Their approach is combinatorial and requires knowledge of the ring structures on both source and target.

Importance

The map is "surjective enough" to conclude that the ring structure on $H_*(\Omega G)$ determines **completely** and **explicitly** the one on $QH^*(\mathcal{O})$.

Theorem (C., a more precise version)

$$\Phi^{\Omega G, \mathcal{O}}_{Savelyev} = \Phi_{Peterson}$$

Thus, our theorem gives a new proof of Peterson's theorem, since we already know $\Phi^{\Omega G, \mathcal{O}}_{Savelyev}$ is a ring homomorphism.

Idea of the computation of $\Phi^{\Omega G, \mathcal{O}}_{Savelyev}$

Theorem (Pressley-Segal)

1. ΩG is a complex manifold.
2. The "universal Hamiltonian fibration" $P(\mathcal{O})$ over ΩG can be constructed in the holomorphic category.

Define

$$\overline{\mathcal{M}} := \text{Deligne-Mumford compactification of holo. sections in } P(\mathcal{O}).$$

We have two evaluation maps $\Omega G \xleftarrow{\text{ev}_1} \overline{\mathcal{M}} \xrightarrow{\text{ev}_2} \mathcal{O}$.

Then $\Phi^{\Omega G, \mathcal{O}}_{Savelyev} = (\text{ev}_2)_* \circ (\text{ev}_1)^*$.

Key lemma

$\overline{\mathcal{M}}$ is smooth.

Idea of the computation of $\Phi^{\Omega G, \mathcal{O}}_{Savelyev}$ (cont.)

Theorem (Fulton-Woodward)

know all two-pointed genus zero GW-invariants for $\mathcal{O}$.

Their arguments:

1. $\overline{\mathcal{M}}_{0,2}(\mathcal{O})$ is smooth + Morse-Smale property for $\langle X, a \rangle$

   $\Longrightarrow$ the space obtained by cutting $\overline{\mathcal{M}}_{0,2}(\mathcal{O})$ with any two Schubert cells is smooth.

2. Observe this space has a T-action

   $\Longrightarrow$ easy to determine its 0-dimensional component.

Idea of the computation of $\Phi^{\Omega G, \mathcal{O}}_{Savelyev}$ (cont.)

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1. ~~$\overline{\mathcal{M}}_{0,2}(\mathcal{O})$~~ $\overline{\mathcal{M}}$ is smooth + Morse-Smale property for $\langle X, a \rangle$ and E

   $\Longrightarrow$ the space obtained by cutting ~~$\overline{\mathcal{M}}_{0,2}(\mathcal{O})$~~ $\overline{\mathcal{M}}$ with ~~any two Schubert cells is smooth.~~ an affine Schubert cell and a Schubert cell is smooth.

2. Observe this space has a T-action

   $\Longrightarrow$ easy to determine its 0-dimensional component.

Thank you!