Hofer geometry of coadjoint orbits and Peterson’s theorem

Chi Hong (Jimmy) Chow

The Chinese University of Hong Kong

5 Nov 2021
Seidel representations

Let \((X, \omega)\) be a compact symplectic manifold.

Denote by \(\text{Ham}(X, \omega)\) the Hamiltonian group of \((X, \omega)\).

Seidel (1997) constructed a group homomorphism

\[
\Phi^X_{\text{Seidel}} : \pi_0(\Omega \text{Ham}(X, \omega)) \rightarrow (\mathbb{Q}H^*(X))^\times
\]

where

- the group structure on \(\pi_0(\Omega \text{Ham}(X, \omega))\) is given by pointwise multiplication in \(\text{Ham}(X, \omega)\),
- \((\mathbb{Q}H^*(X))^\times\) is the multiplicative subgroup of invertible elements of \(\mathbb{Q}H^*(X)\).
The construction

\( f \in \Omega \text{Ham}(X, \omega) \leadsto P_f(X) := C \times X \cup C \times X \left/ (z, x) \sim (z^{-1}, f(\frac{z}{|z|}) \cdot x) \right. \)

\[ \begin{array}{c}
\uparrow \\
C P^1 := C \cup C \left/ z \sim z^{-1} \right.
\end{array} \]

Known: \( P_f(X) \) is a Hamiltonian fibration over \( \mathbb{C}P^1 \) with fibers \((X, \omega)\).

Definition

\[ \Phi^X_{Seidel}([f]) := \sum_i \# \left\{ \text{holo. section in } P_f(X) \right\} e^i q^{\text{cont.by.holo.sect.}} \]

where \( \{ e_i \}, \{ e^i \} \) are dual bases of \( H^*(X) \).

A gluing argument \( \implies \Phi^X_{Seidel} \) is a group homomorphism.
A parametrized version

Savelyev (2008) defined a ring map extending Seidel’s map

\[ \Phi_{\text{Savelyev}}^X : H^{-\ast}(\Omega \text{Ham}(X, \omega)) \to QH^{\ast}(X) \]

\[ f : \Gamma \to \Omega \text{Ham}(X, \omega) \leadsto P_f(X) := \mathbb{C} \times \Gamma \times X \cup \mathbb{C} \times \Gamma \times X / (z, \gamma, x) \sim (z^{-1}, \gamma, f_\gamma(\frac{z}{|z|}) \cdot x) \]

\[ \downarrow \]

\[ \mathbb{C}P^1 \times \Gamma := \mathbb{C} \times \Gamma \cup \mathbb{C} \times \Gamma / (z, \gamma) \sim (z^{-1}, \gamma) \]

\( P_f(X) \) can be considered as a smooth family \( \{ P_{f_\gamma}(X) \}_{\gamma \in \Gamma} \) of Hamiltonian fibrations parametrized by \( \Gamma \).

Definition

\[ \Phi_{\text{Savelyev}}^X([f]) := \sum_i \# \left\{ \left( \gamma, \text{holo. section in } P_{f_\gamma}(X) \right) \right\} e^{i q_{\text{cont.by.holo.sect.}}} \]

A gluing argument \( \Longrightarrow \Phi_{\text{Savelyev}}^X \) is a ring homomorphism.
Main theorem

Let $G$ be a compact Lie group and $\mathcal{O}$ a coadjoint orbit of $G$.

Example

$G := SU(n)$. All coadjoint orbits of $G$ are of the form

$$F\ell(k_1, \ldots, k_r; n) := \{0 \subset V_1 \subset \cdots \subset V_r \subset \mathbb{C}^n | \dim V_i = k_i\}$$

where $0 < k_1 < \cdots < k_r < n$ are integers.

Define $\Phi_{\Omega G, \mathcal{O}}^{\text{Savelyev}}$ to be the composition

$$H_{-\ast}(\Omega G) \xrightarrow{\text{natural map}} H_{-\ast}(\Omega \text{Ham}(\mathcal{O})) \xrightarrow{\Phi_{\mathcal{O}}^{\text{Savelyev}}} QH^\ast(\mathcal{O}).$$

Theorem (C.)

Know $\Phi_{\Omega G, \mathcal{O}}^{\text{Savelyev}}$ completely.
Basis of $H^*(O)$: Schubert classes

Represented by **Schubert cells** = descending manifolds wrt the Morse function $f(X) := \langle X, a \rangle$ on $O$ where $a \in \text{Lie}(G)^\vee$ is an generic element.

**Example**

$G := SU(3), O := \mathbb{CP}^2$

Take a maximal torus $T$ e.g. $T := \{\text{diagonal matrices}\}$.
Assume $a \in \text{Lie}(T)$ generic. Then $\text{Crit}(f) = \text{Lie}(T) \cap O = \{\bullet\}$.
Basis of $H_*(\Omega G)$: Affine Schubert classes

Represented by **affine Schubert cells** = descending manifolds wrt a perturbation of the energy functional $E$ on $\Omega G$. Notice $E$ is Bott-Morse whose critical set = a countable disjoint union of coadjoint orbits of $G$.

**Example**

$G \coloneqq SU(3)$. Define $\{\bullet\} \coloneqq \exp^{-1}(e) \cap \text{Lie}(T)$, the unit lattice of $\text{Lie}(T)$.

Every $\mu \in \{\bullet\}$ gives rise to an affine Schubert class $\xi_\mu$. 

\[
\begin{array}{c}
\mu_0 \\
\mu_1 \\
\mu_2
\end{array}
\]

\[
\begin{array}{c}
\mu_0 \\
\xi_{\mu_0} \\
\mu_1 \\
\mu_2
\end{array}
\]
Basis of $H_*(\Omega G)$: Affine Schubert classes

Represented by **affine Schubert cells** = descending manifolds wrt a perturbation of the energy functional $E$ on $\Omega G$. Notice $E$ is Bott-Morse whose critical set = a countable disjoint union of coadjoint orbits of $G$.

**Example**

$G := SU(3)$. Define $\{\bullet\} := \exp^{-1}(e) \cap \text{Lie}(T)$, the unit lattice of $\text{Lie}(T)$.

Every $\mu \in \{\bullet\}$ gives rise to an affine Schubert class $\xi_{\mu}$.
Basis of $H_*(\Omega G)$: Affine Schubert classes

Represented by **affine Schubert cells** = descending manifolds wrt a perturbation of the energy functional $E$ on $\Omega G$.

Notice $E$ is Bott-Morse whose critical set = a countable disjoint union of coadjoint orbits of $G$.

**Example**

$G := SU(3)$. Define $\{\bullet\} := \exp^{-1}(e) \cap \text{Lie}(T)$, the unit lattice of $\text{Lie}(T)$.

Every $\mu \in \{\bullet\}$ gives rise to an affine Schubert class $\xi_\mu$. 

Every $\mu \in \{\bullet\}$ gives rise to an affine Schubert class $\xi_\mu$.

\[ \mathbb{CP}^2 \]
Roughly, it sends a basis element to zero or an explicit basis element.

Example

\[ G := SU(3), O := \mathbb{C}P^2 \]

For simplicity, set \( q = 1 \) (\( q \) = quantum variable).
Consequence 1: Non-triviality of $\pi_*(\text{Ham}(\mathcal{O})) \otimes \mathbb{Q}$

Theorem
The dimension of the kernel of the induced map
$$\pi_*(G) \otimes \mathbb{Q} \rightarrow \pi_*(\text{Ham}(\mathcal{O})) \otimes \mathbb{Q}$$
is at most the number of facets of a Weyl chamber passing through a nearby point $\in \text{Lie}(T) \cap \mathcal{O}$.

Example
This upper bound is equal to $n - 1 - r$ for the case $\mathcal{O} = F\ell(k_1, \ldots, k_r; n)$. Notice $\dim \pi_*(\text{SU}(n)) \otimes \mathbb{Q} = n - 1$.

Corollary
The induced map is injective if $\mathcal{O}$ is generic, e.g. $F\ell(1, 2, \ldots, n - 1; n)$.

Remark
For generic $\mathcal{O}$, Kędra proved a much stronger result based on the work of Reznikov, Kędra-McDuff, Gal-Kędra-Tralle:

$$H^*(B\text{Homeo}(\mathcal{O}); \mathbb{Q}) \rightarrow H^*(BG; \mathbb{Q})$$
is surjective.
Consequence 2: Hofer geometry of $Ham(\mathcal{O})$

Let $(X, \omega)$ be a compact symplectic manifold.
Let $\{\varphi_t\}$ be a path in $Ham(X, \omega)$.
There exists a unique family $\{H_t : X \to \mathbb{R}\}$, called the normalized generating Hamiltonian of $\{\varphi_t\}$, satisfying

$$
\begin{align*}
\dot{\varphi}_t &= X_{H_t} \circ \varphi_t \\
\int_X H_t \omega^{\frac{1}{2} \dim X} &= 0
\end{align*}
$$

Define the positive Hofer length functional $L^+$ on $\Omega Ham(X, \omega)$

$$
L^+(\{\varphi_t\}) := \int_0^1 \max_X H_t \, dt.
$$

A variational problem

Given a homology class $A \in H_*(\Omega Ham(X, \omega))$, minimize

$$
\max_{\Gamma} L^+ \circ f
$$

over all smooth cycles $f : \Gamma \to \Omega Ham(X, \omega)$ representing $A$. 
**Consequence 2: Hofer geometry of $Ham(\mathcal{O})$ (cont.)**

**Theorem**
For any $\mu \in \{\bullet, \circ, \cdot\}$, there exists a constant $C_\mu$ such that for any smooth cycle $f : \Gamma \to \Omega Ham(\mathcal{O})$ representing $\xi_\mu$,\[
\max_{\Gamma} L^+ \circ f \geq C_\mu.
\]

Moreover, $C_\mu$ is attained by an explicit cycle.

The proof uses the computation of $\Phi_{\text{Savelyev}}^{\Omega G, \mathcal{O}}$ and a standard argument, e.g. Akveld-Salamon/ McDuff-Slimowitz/ McDuff-Tolman/ Savelyev etc.

**Remark**
Savelyev computed $\Phi_{\text{Savelyev}}^{\Omega G, \mathcal{O}}$ up to higher action terms when $\mathcal{O}$ is generic and $\mu \in$ the interior of the anti-dominant chamber. This suffices to prove the above theorem for this case.
Consequence 3: A new proof of Peterson’s theorem

Theorem (Peterson 1997 unpublished, Lam-Shimozono 2010)
An explicit linear map
\[ \Phi_{Peterson} : H_{-*}(\Omega G) \to QH^*(\mathcal{O}) \]
is a ring homomorphism.
Their approach is combinatorial and requires knowledge of the ring structures on both source and target.

Importance
The map is “surjective enough” to conclude that the ring structure on \( H_{*}(\Omega G) \) determines completely and explicitly the one on \( QH^{*}(\mathcal{O}) \).

Theorem (C., a more precise version)
\[ \Phi_{\Omega G,\mathcal{O}}^{Savelyev} = \Phi_{Peterson} \]
Thus, our theorem gives a new proof of Peterson’s theorem, since we already know \( \Phi_{\Omega G,\mathcal{O}}^{Savelyev} \) is a ring homomorphism.
Idea of the computation of $\Phi_{\Omega G, \mathcal{O}}^{Savelyev}$

**Theorem (Pressley-Segal)**

1. $\Omega G$ is a complex manifold.
2. The “universal Hamiltonian fibration” $P(\mathcal{O})$ over $\Omega G$ can be constructed in the holomorphic category.

Define

$$\overline{M} := \text{Deligne-Mumford compactification of holo. sections in } P(\mathcal{O}).$$

We have two evaluation maps $\Omega G \xleftarrow{\text{ev}_1} \overline{M} \xrightarrow{\text{ev}_2} \mathcal{O}$.

Then $\Phi_{\Omega G, \mathcal{O}}^{Savelyev} = (\text{ev}_2)^* \circ (\text{ev}_1)^*$.

**Key lemma**

$\overline{M}$ is smooth.
Idea of the computation of $\Phi^{\Omega G, \mathcal{O}}_{Savel'yev}$ (cont.)

Theorem (Fulton-Woodward)

know all two-pointed genus zero GW-invariants for $\mathcal{O}$.

Their arguments:

1. $\overline{M}_{0, 2}(\mathcal{O})$ is smooth + Morse-Smale property for $\langle X, a \rangle$

   $\implies$ the space obtained by cutting $\overline{M}_{0, 2}(\mathcal{O})$ with any two Schubert cells is smooth.

2. Observe this space has a $T$-action

   $\implies$ easy to determine its 0-dimensional component.
Idea of the computation of $\Phi_{\text{Savelyev}}^{\Omega G, \mathcal{O}}$ (cont.)

Theorem (Fulton-Woodward)
know all two-pointed genus zero GW-invariants for $\mathcal{O}$.

Their arguments: Our arguments:

1. $\overline{\mathcal{M}}_{0,2}(\mathcal{O})$ is smooth + Morse-Smale property for $\langle X, a \rangle$ and $E$

   $\implies$ the space obtained by cutting $\overline{\mathcal{M}}_{0,2}(\mathcal{O})$ with any two Schubert cells is smooth. an affine Schubert cell and a Schubert cell is smooth.

2. Observe this space has a $T$-action

   $\implies$ easy to determine its 0-dimensional component.
Thank you!