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WITTEN'S COMPLEX AND INFINITE DIMENSIONAL MORSE THEORY

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The fact that the trajectory spaces are framed rather than only oriented suggests the following extension of the above program: If h^* denotes a general cohomology theory, then it should be possible to obtain $h^*(I(S))$ in a way similar to that above through an analysis of trajectory spaces. The chain complex would have to be replaced by $h^*(I(C)) = \bigoplus_{x \in C} h^{*-\mu(x)}$, and the δ-homomorphism would, in contrast to the singular case, depend on

trajectory spaces of arbitrary dimension. For example, in the case of stable cohomotopy $h^* = \pi_s^*$, the contribution of every compact component of $\hat{M}(x,y)$ should be given by the element of π_s^* classifying its framed cobordism type. (In fact, the spaces $\hat{M}(x,y)$ undergo framed cobordisms under a change of the metric.) Of course, the higher dimensional components of $\hat{M}(x,y)$ do not have to be compact, but a reasonable modification of the program should lead to a spectral sequence converging to $h^*(I(S))$, as one would expect. This program is only of limited use for finite dimensional Morse theory, but might have applications to infinite dimensional cases.

Complex cobordism and Hamiltonian fibrations

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Hamiltonian diffeomorphisms and self-homotopy equivalences

Let $\mathrm{Ham}(X)$ denote the Hamiltonian diffeomorphism group of a closed symplectic manifold X. We have natural maps

$$\mathrm{Ham}(X) \to \mathrm{Diff}(X) \to \mathrm{Homeo}(X) \to \mathrm{Aut}(X)$$

At each of these steps, we are throwing away more and more information about X, but we still have a lot of structure, e.g. the ring structure on the cohomology of X is preserved.
We can go further, and pass from $\mathrm{Aut}(X)$ to the space of *stable auto-equivalences*, $\mathrm{SAut}(X)$, obtained by replacing X with $\Sigma^n X = S^n \times X/\{\infty\} \times X$. In this last step, we only retain the additive structure on the (co)-homology of X:

$$H_* \, \mathrm{SAut}(X) \to \mathrm{End}_*(H_* X).$$

We shall be interested in the associated map on loop spaces:

Conjecture

If $\mathbb{E}$ is a complex-oriented cohomology theory, then the map $\Omega \, \mathrm{Ham}(X) \to \Omega \, \mathrm{SAut}(X)$ vanishes with $\mathbb{E}$-coefficient, at the chain level. In particular, this map vanishes on $\mathbb{E}$-homology, and so does the map to the (higher) endomorphisms of the $\mathbb{E}$-homology of X.

Complex-oriented (co)homology theories

A homology theory $H_*(\_;\mathbb{E})$ is a functor from spaces to abelian groups, with the property that the maps induced by homotopy equivalences are isomorphisms, so that the Mayer-Vietoris sequence for a (reasonable) cover is exact, and which is equipped with a natural isomorphism $\tilde{H}_{n+*}(\Sigma^n X, \mathbb{E}) \cong H_*(X, \mathbb{E})$. With further mild assumptions, one can obtain associated *cohomology groups* $H^*(\_;\mathbb{E})$, as well as compactly supported cohomology groups $H^*_c(\_;\mathbb{E})$.

Definition

A (co)-homology theory $\mathbb{E}$ is complex oriented if there is a natural Poincaré duality isomorphism for stably almost complex manifolds of dimension n:

$$H^*_c(M;\mathbb{E}) \cong H_{n-*}(M;\mathbb{E}).$$

Results from the 60's/70's show that complex cobordism is the universal complex oriented cohomology theory.

(Co)-homology theories and spectra

By now, a standard idea in Floer theory is that there is more information at the chain level than after passing to homology.
The same principle holds in the setting of generalised (co)-homology theories. Associated to $\mathbb{E}$, are a collection of *topological spaces* $\mathbb{E}(n)$, and we have:

$$H^*(X; \mathbb{E}) \equiv \pi_*(\operatorname{colim}_n \operatorname{Map}(\Sigma^n X, \mathbb{E}(n)),$$

where Map is the space of continuous maps, and π_* stands for homotopy groups (there is a similar formula for homology). Showing that a map vanishes *at the chain level* means finding null-homotopies at the level of these mapping spaces (rather than showing that the maps vanish on homotopy groups).

Remark

A fundamental idea to have in mind is that showing that some map on a (co)-bordism theory vanishes at the chain level entails finding bounding manifolds, whereas showing that it vanishes on (co)-homology is about proving that a manifold bounds.

Current results

We do not yet have a proof of the conjecture on the first slide, but our main result is about the analogous statement for elements of the loop space:

Theorem (A-McLean-Smith '21)

If $\mathbb{E}$ is a complex-oriented cohomology theory, then, for each loop $S^1 \to \mathrm{Ham}(X)$, the composite map $S^1 \to \mathrm{SAut}(X)$ vanishes with $\mathbb{E}$-coefficient, at the spectrum level. In particular, this map vanishes on $\mathbb{E}$-homology, and so does the map to the automorphisms of the $\mathbb{E}$-homology of X.

Since $\mathrm{Ham}(X)$ is (by definition) connected, this is the first statement that one can try to prove.
If $\mathbb{E}$ is rational cohomology, this result is due to Lalonde-McDuff-Polterovich, using the Seidel homomorphism (they work in the monotone case, and McDuff later worked out the virtual cycle theory to remove this assumption).

Example

The "rotation" loop $S^1 \to \mathrm{Ham}(S^2)$ is non-trivial on KO-theory.

Consequences for Hamiltonian fibrations

A loop $\phi\colon S^1 \to \mathrm{Ham}(X)$ determines, by a clutching construction, a Hamiltonian fibration $X \to P_\phi \to S^2$.

Corollary

If $\mathbb{E}$ is a complex-oriented cohomology theory, then the Serre spectral sequence degenerates, and we have an additive isomorphism

$$H_*(P_\phi; \mathbb{E}) \cong H_*(X; \mathbb{E}) \oplus H_{*-2}(X; \mathbb{E}).$$

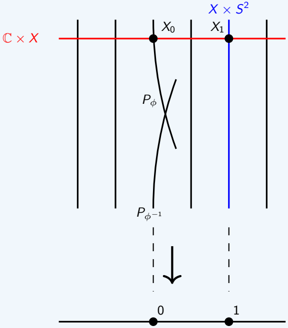
Again, this comes from a chain-level result, asserting that the fibration $P_\phi \to S^2$ becomes trivial after passing to $\mathbb{E}$-chains. The chain level result implies:

Corollary

If $\mathbb{E} = \mathbb{F}_p$ then the above isomorphism respects the subalgebra of Steenrod operations generated by the Milnor Q_i elements.

A simple normal crossings degeneration

In his work introducing a homomorphism from $\pi_1 \operatorname{Ham}(X)$ to the quantum cohomology of X, Seidel related holomorphic sections in P_ϕ and $P_{\phi^{-1}}$ to curves in the trivial fibration $X \times S^2$ via a gluing argument. McDuff reinterpreted Seidel's construction: there is a symplectic manifold $\tilde{P}$, equipped with a map to $\mathbb{C}$ so that: (i) the unique singular fibre is the union of P_ϕ and $P_{\phi^{-1}}$ along X, and (ii) the fibres away from 0 are $X \times S^2$.



Constructing a splitting

Proving that Serre spectral sequence for $P_\phi \to S^2$ degenerates amounts to constructing a splitting of the map $X \to P_\phi$ on homology. Composing with the map $P_\phi \to \tilde{P}$, it suffices to split the composite

$$H_*(X;\mathbb{E}) \to H_*(\tilde{P};\mathbb{E}).$$

The advantage of $\tilde{P}$ over P_ϕ is that there is a canonical class $\beta \in \pi_2(\tilde{P})$ associated to the (trivial) sections of the fibre at 1. Let $\overline{\mathcal{M}}_{2;\beta}(\tilde{P}, \mathbb{C} \times X)$ denote the moduli space of genus 0 stable maps in $\tilde{P}$ with 2 marked points, one of which is constrained to lie on the *horizontal slice*. We have

$$H_*(\tilde{P};\mathbb{E}) \cong H^*_c(\tilde{P};\mathbb{E}) \to H^*_c(\overline{\mathcal{M}}_{2;\beta}(\tilde{P},\mathbb{C}\times X);\mathbb{E}) \to H_*(\overline{\mathcal{M}}_{2;\beta}(\tilde{P},\mathbb{C}\times X);\mathbb{E}) \to H_*(\mathbb{C}\times X;\mathbb{E}) \cong H_*(X;\mathbb{E}).$$

The middle arrow is given by capping against the virtual fundamental class of $\overline{\mathcal{M}}_{2;\beta}(\tilde{P}, \mathbb{C} \times X)$. As long as this virtual fundamental class satisfies reasonable properties, one obtains the desired splitting.

Remark

We do not use quantum cohomology, unlike Lalonde-McDuff-Polterovich. At the moment, we don't have a theory of quantum cohomology which is adequate for this task. Our proof is simpler because there is no gluing!

Existence of global charts

Symplectic topology now has many approaches to VFC. With the exception of polyfolds and Cieliebak-Mohnke's approach, they all rely on local-to-global arguments. We develop a new approach which proves:

Theorem (A-McLean-Smith)

The moduli space $\overline{\mathcal{M}}_n(X;\beta)$ of stable genus 0 maps in X in a fixed class β admits a global Kuranishi chart: there is a compact Lie group G, a smooth almost complex G-manifold $\mathcal{T}$, a complex G-vector bundle E over $\mathcal{T}$, and a section s of E, so that

$$\overline{\mathcal{M}}_n(X;\beta) \cong s^{-1}(0)/G.$$

This is unique up to (i) shrinking $\mathcal{T}$, (ii) stabilising by a vector bundle on $\mathcal{T}$, and (iii) enlarging G.

In our construction, $G = U(d)$ for some large value of d depending on β. The smooth structure is constructed indirectly, starting with the smooth structure on $\overline{\mathcal{M}}(\mathbb{CP}^{d-1})$, and using smoothing theory.

Basic construction

Pick a line bundle $\mathcal{L}$ on M with curvature sufficiently close to ω. Choosing k large enough, we find that $\mathcal{L}^{\otimes k}$ pulls back to a very ample line bundle on every non-constant component of a curve in $\overline{\mathcal{M}}_n(X;\beta)$. Define a *framed map* to be a stable map $u\colon \Sigma \to X$, equipped with a (unitary) basis of holomorphic sections of $u^*\mathcal{L}^{\otimes k}$. We have the following key facts:

1. The space of framed maps has a natural action of $U(d)$, with quotient $\overline{\mathcal{M}}_n(X;\beta)$.
2. The space of framed maps projects to $\overline{\mathcal{M}}_d(\mathbb{CP}^{d-1})$, with image lying in the manifold locus.
3. The domain of a framed map is identified with the fibre of the universal curve over $\overline{\mathcal{M}}_d(\mathbb{CP}^{d-1})$.

The last part should essentially be interpreted as saying that the datum of the framing stabilises the domain of a stable map. This is usually the essential problem in achieving transversality. In particular, choosing inhomogeneous data parametrised by the universal curve over $\overline{\mathcal{M}}_d(\mathbb{CP}^{d-1})$, we may thicken the space of framed maps to a $U(d)$-manifold $\mathcal{T}$, carrying an equivariant vector bundle with a section whose 0-locus is the space of framed maps.

Remark

This construction should be seen as a soft version of Cieliebak-Mohnke's. We start with the same ingredient (a hermitian line bundle on X), but we never pick a Donaldson section (breaking symmetry).

The smooth structure

The work of Mazur, Hirsh, Kirby-Siebenmann, ... provides a criterion for when a topological manifold admits a smooth structure: it suffices to lift the tangent microbundle (with structure group $\mathrm{Homeo}(\mathbb{R}^N, 0)$) to a vector bundle.
Since $\overline{\mathcal{M}}_d(\mathbb{CP}^{d-1})$ is a smooth Deligne-Mumford stack, there is no ambiguity in constructing a smooth structure on it. On the other hand, the fibres of $\mathcal{T}$ over $\overline{\mathcal{M}}_d(\mathbb{CP}^{d-1})$ are naturally smooth because the domain is fixed. In fact, the strata of $\mathcal{T}$ corresponding to a fixed topological type are smooth.
Gluing theory provides charts in the normal direction of each stratum. These are compatible with the smooth structures of the fibres, so we obtain a *fibrewise smooth structure* (analogous to a bundle of smooth manifolds). We obtain a *fibrewise tangent bundle* on $\mathcal{T}$, and the direct sum with the pullback of the tangent space of $\overline{\mathcal{M}}_d(\mathbb{CP}^{d-1})$ is a lift to the tangent microbundle.

Theorem (Lashof)

A topological G-manifold with an equivariant lift of the tangent microbundle to a vector bundle admits a smooth structure after stabilisation whenever there are only finitely many orbit types.

The virtual fundamental class

By shrinking the thickening, we may assume that it is a manifold with boundary. The most naive statement of equivariant Poincaré duality to formulate in this context is an isomorphism

$$H^*_G(\mathcal{T}; \mathbb{E}) \cong H_*^G(\mathcal{T}, \partial\mathcal{T}; \mathbb{E}),$$

so we obtain a G-equivariant fundamental class $[\mathcal{T}]_G \in H_*^G(\mathcal{T}, \partial\mathcal{T}; \mathbb{E})$. The section s determines a G-equivariant Euler class $[\eta_E]_G \in H^*_G(\mathcal{T}, \partial\mathcal{T}; \mathbb{E})$.

Definition

The (image in $\mathcal{T}$ of the) equivariant fundamental class $[s^{-1}(0)]_G$ is the image of $[\eta_E]_G \otimes [\mathcal{T}]_G$ under the cap product

$$H^*_G(\mathcal{T}, \partial\mathcal{T}; \mathbb{E}) \otimes H_*^G(\mathcal{T}, \partial\mathcal{T}; \mathbb{E}) \to H_*^G(\mathcal{T}; \mathbb{E}).$$

It is technically difficult to arrange for this class to live on the moduli space itself, but easy to relate the classes associated to different thickenings. In particular, the image under evaluation maps is independent of choice.

$K(n)$-local cohomology theories

From the above, we see that the key missing ingredient is the isomorphism

$$H^*_G(\mathcal{T};\mathbb{E}) \cong H^G_*(\mathcal{T},\partial\mathcal{T};\mathbb{E}).$$

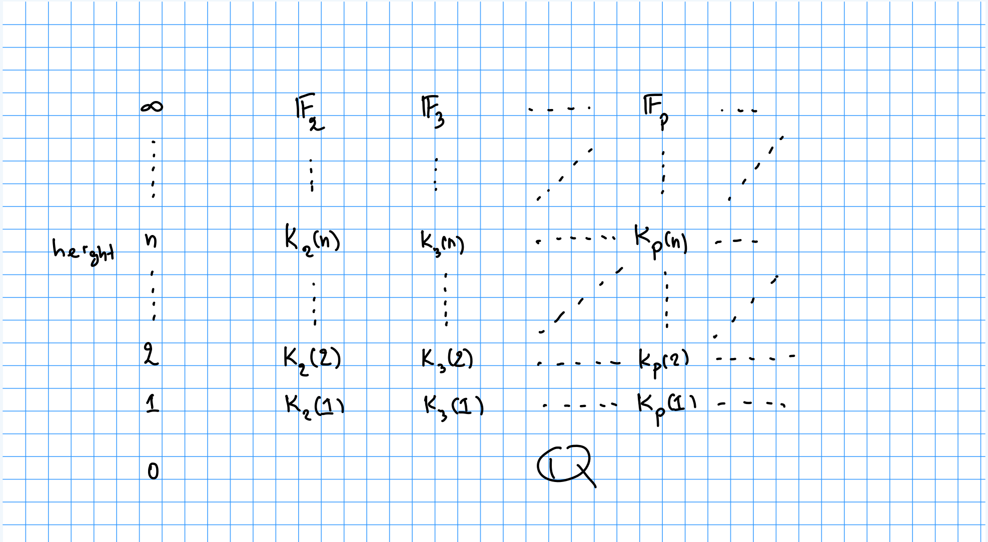
If $\mathbb{E}$ is the ordinary cohomology theory associated to a ring $\mathbf{k}$, this only holds if the cardinality of the isotropy groups is relatively prime to the characteristic of $\mathbf{k}$.

Theorem (Cheng)

If $\mathbb{E}$ is a $K(n)$-local cohomology theory (which is complex oriented), then almost complex G manifolds (with finite isotropy) satisfy Poincaré duality in this sense.

The key case is that of a point with the action of a finite group Γ, in which case the result goes back to Greenlees and Sadofsky, who derived it from a computation by Wilson of the (co)-homology of $B\Gamma$.

The decomposition of the stable category into chromatic primes



Transchromatic induction

A basic result of commutative algebra, is that a map of finitely generated chain complexes which is null-homotopic after p-completion is null-homotopic. In homotopy theory, there is the following analogue:

Lemma

If X is a finite CW-complex, then a map $X \to Y$ is trivial with respect to a homology theory $\mathbb{E}$ (at the spectrum level) if and only if it is trivial with respect to the p-completion of $\mathbb{E}$ (for all ordinary primes p).

In order to pass from a result about $K(n)$-local theories to one about complex cobordism, we also use the following surprising result:

Theorem (Hovey)

The p-completion of MU is a wedge summand of the product of the localisations of MU at the Morava K-theories of all heights.