Estimating Reeb Chords Using Microlocal Sheaf Theory

Symplectic Zoominar

Wenyuan Li

Northwestern
Consider the contact manifold $J^1(M) = T^*M \times \mathbb{R}_t$, with contact form $\alpha_{\text{std}} = dt - \sum_{i=1}^{n} \xi_i dx_i$. The Reeb vector field of $(J^1(M), \alpha_{\text{std}})$ is $R_\alpha = \partial_t$. Hence Reeb chords on $\Lambda$ correspond to the immersed points in the Lagrangian projection.
Consider the contact manifold $J^1(M) = T^*M \times \mathbb{R}_t$, with contact form $\alpha_{std} = dt - \sum_{i=1}^n \xi_i dx_i$.

A Legendrian submanifold $\Lambda \subset J^1(M)$ satisfies $\alpha_{std}|_{\Lambda} = 0$. The front projection is $\pi : J^1(M) \to M \times \mathbb{R}$, while the Lagrangian projection is $\pi_{Lag} : J^1(M) \to T^*M$. 
Consider the contact manifold $J^1(M) = T^* M \times \mathbb{R}_t$, with contact form $\alpha_{\text{std}} = dt - \sum_{i=1}^n \xi_i dx_i$.

A Legendrian submanifold $\Lambda \subset J^1(M)$ satisfies $\alpha_{\text{std}}|\Lambda = 0$. The front projection is $\pi : J^1(M) \to M \times \mathbb{R}$, while the Lagrangian projection is $\pi_{\text{Lag}} : J^1(M) \to T^* M$.

The Reeb vector field of $(J^1(M), \alpha_{\text{std}})$ is $R_\alpha = \partial_t$. Hence Reeb chords on $\Lambda$ correspond to the immersed points in the Lagrangian projection.
For a sheaf $\mathcal{F}$ on $X$, its singular support $SS^\infty(\mathcal{F}) \subset T^*,\infty X$ encodes “the points and codirections where the stalk of the sheaf jumps”. It is always Legendrian or coisotropic (Kashiwara-Schapira).
For a sheaf $\mathcal{F}$ on $X$, its singular support $SS^\infty(\mathcal{F}) \subset T^*\times X$ encodes "the points and codirections where the stalk of the sheaf jumps". It is always Legendrian or coisotropic (Kashiwara-Schapira).

The microstalk of $\mathcal{F}$ at $(x, \xi)$ is "the difference between the stalk before/after passing $x$ along the codirection $\xi$".
For a sheaf $\mathcal{F}$ on $X$, its singular support $SS^\infty(\mathcal{F}) \subset T^*,\infty X$ encodes “the points and codirections where the stalk of the sheaf jumps”. It is always Legendrian or coisotropic (Kashiwara-Schapira).

The microstalk of $\mathcal{F}$ at $(x, \xi)$ is ”the difference between the stalk before/after passing $x$ along the codirection $\xi$”.

**Theorem (Guillermou-Kashiwara-Schapira 12’)**

The category $Sh^b_\Lambda(X)$ of sheaves with $SS^\infty(\mathcal{F}) \subset \Lambda$ is invariant under Legendrian isotopies.
Viewing $\Lambda \subset J^1(M)$ as in $T^{\ast,\infty}(M \times \mathbb{R})$ (via $(x, \xi, t) \mapsto (x, \xi, t, 1)$), one can consider sheaves with singular support on $\Lambda$. 

$\begin{align*}
M &= \text{pt} \\
M &= \mathbb{R} \\
M &= S^1
\end{align*}$
Microlocal Sheaves

- Viewing $\Lambda \subset J^1(M)$ as in $T^{*,\infty}(M \times \mathbb{R})$ (via $(x, \xi, t) \mapsto (x, \xi, t, 1)$), one can consider sheaves with singular support on $\Lambda$.
- Sheaves in $\mathcal{S}h^b_\Lambda(M \times \mathbb{R})$ are sheaves that are locally constant on each smooth stratum of the front $\pi(\Lambda)$ (plus extra conditions).

\[ M = \text{pt} \quad M = \mathbb{R} \quad M = S^1 \]
Consider the self Reeb chords on a Legendrian $\Lambda \subset J^1(M)$. This is just the immersed points of $\pi_{\text{Lag}}(\Lambda)$.
Consider the self Reeb chords on a Legendrian $\Lambda \subset J^1(M)$. This is just the immersed points of $\pi_{\text{Lag}}(\Lambda)$.

For self Reeb chords of $\Lambda$, the Arnol'd type bound one would expect in good situations should be $\frac{1}{2} \sum_{i=0}^{n} b_i(\Lambda)$. 
Consider the self Reeb chords on a Legendrian $\Lambda \subset J^1(M)$. This is just the immersed points of $\pi_{\text{Lag}}(\Lambda)$.

For self Reeb chords of $\Lambda$, the Arnol’d type bound one would expect in good situations should be $\frac{1}{2} \sum_{i=0}^{n} b_i(\Lambda)$.

Let $Q(\Lambda)$ (resp. $Q_i(\Lambda)$) be the set of all (resp. degree $i$) Reeb chords on $\Lambda$. Suppose $\pi_{\text{Lag}}(\Lambda)$ is immersed with transverse double points.
Main Results I

Theorem (L.)

Let $M$ be orientable, $\Lambda \subset J^1(M)$ be a closed Legendrian and $k$ be a field (and $\Lambda$ is spin when $\text{char} k \neq 2$).

1. If there exists a $k$-coefficient sheaf $F \in \mathbf{Sh}_\Lambda(M \times \mathbb{R})$ with microlocal rank $r$ such that $\text{supp}(F)$ is compact, then $|Q_i(\Lambda)| + |Q_{n-i}(\Lambda)| \geq b_i(\Lambda; k)$.

2. If there exists a $k$-coefficient sheaf $F \in \mathbf{Sh}_\Lambda(M \times \mathbb{R})$ with perfect (micro)stalk such that $\text{supp}(F)$ is compact, then $|Q(\Lambda)| \geq \frac{1}{2} n \sum_{i=0}^{n} b_i(\Lambda; k)$. 
Main Results I

Theorem (L.)

Let $M$ be orientable, $\Lambda \subset J^1(M)$ be a closed Legendrian and $k$ be a field (and $\Lambda$ is spin when $\text{char } k \neq 2$).

1. If there exists a $k$-coefficient sheaf $\mathcal{F} \in \text{Sh}^b_\Lambda(M \times \mathbb{R})$ with microlocal rank $r$ such that $\text{supp}(\mathcal{F})$ is compact, then

$$|Q_i(\Lambda)| + |Q_{n-i}(\Lambda)| \geq b_i(\Lambda; k).$$
Theorem (L.)

Let $M$ be orientable, $\Lambda \subset J^1(M)$ be a closed Legendrian and $k$ be a field (and $\Lambda$ is spin when $\text{char } k \neq 2$).

1. If there exists a $k$-coefficient sheaf $\mathcal{F} \in Sh^b_{\Lambda}(M \times \mathbb{R})$ with microlocal rank $r$ such that $\text{supp}(\mathcal{F})$ is compact, then

$$|Q_i(\Lambda)| + |Q_{n-i}(\Lambda)| \geq b_i(\Lambda; k).$$

2. If there exists a $k$-coefficient sheaf $\mathcal{F} \in Sh^b_{\Lambda}(M \times \mathbb{R})$ with perfect (micro)stalk such that $\text{supp}(\mathcal{F})$ is compact, then

$$|Q(\Lambda)| \geq \frac{1}{2} \sum_{i=0}^{n} b_i(\Lambda; k).$$
Conjecturally, microlocal rank $r$ sheaves are equivalent to $r$-dimensional representations of the Legendrian contact homology (when $r = 1$ they are called augmentations).
Conjecturally, microlocal rank $r$ sheaves are equivalent to $r$-dimensional representations of the Legendrian contact homology (when $r = 1$ they are called augmentations).

Ekholm-Etnyre-Sullivan, Ekholm-Etnyre-Sabloff, & Dimitroglou & Rizell-Golovko proved the same bound when $\mathcal{LCH}^*(\Lambda)$ has an $r$-dimensional representation and $\Lambda$ is horizontally displaceable. Sabloff-Traynor proved the bound when $\Lambda$ has a generating family linear at infinity.

We can show that the existence of a sheaf plus horizontal displaceability implies compact support of the sheaf, but the converse is false. On the other hand, existence of generating families linear at infinity probably implies existence of sheaves over any ring, but there are Legendrians that only admit sheaves over certain rings.
Conjecturally, microlocal rank $r$ sheaves are equivalent to $r$-dimensional representations of the Legendrian contact homology (when $r = 1$ they are called augmentations).

Ekholm-Etnyre-Sullivan, Ekholm-Etnyre-Sabloff, & Dimitroglou Rizell-Golovko proved the same bound when $LCH_\ast(\Lambda)$ has an $r$-dimensional representation and $\Lambda$ is horizontally displaceable. Sabloff-Traynor proved the bound when $\Lambda$ has a generating family linear at infinity.

We can show that the existence of a sheaf plus horizontal displaceability implies compact support of the sheaf, but the converse is false. On the other hand, existence of generating families linear at infinity probably implies existence of sheaves over any ring, but there are Legendrians that only admit sheaves over certain rings.
Consider the Reeb chords between $\Lambda$ and $\varphi^1_H(\Lambda)$. When $H$ can be lifted from a symplectic Hamiltonian $H_{\text{symp}}$, this is just the Lagrangian intersection between $\pi_{\text{Lag}}(\Lambda)$ and $\varphi^1_{H_{\text{symp}}}(\pi_{\text{Lag}}(\Lambda))$.
Consider the Reeb chords between $\Lambda$ and $\varphi_H^1(\Lambda)$. When $H$ can be lifted from a symplectic Hamiltonian $H_{\text{symp}}$, this is just the Lagrangian intersection between $\pi_{\text{Lag}}(\Lambda)$ and $\varphi_{H_{\text{symp}}}^1(\pi_{\text{Lag}}(\Lambda))$.

Define the oscillation norm of the Hamiltonian to be

$$\|H_s\|_{osc} = \int_0^1 \left( \max_{x \in P \times \mathbb{R}} H_s - \min_{x \in P \times \mathbb{R}} H_s \right) ds.$$
Main Result II

- Consider the Reeb chords between $\Lambda$ and $\varphi^1_H(\Lambda)$. When $H$ can be lifted from a symplectic Hamiltonian $H_{symp}$, this is just the Lagrangian intersection between $\pi_{Lag}(\Lambda)$ and $\varphi^1_{H_{symp}}(\pi_{Lag}(\Lambda))$.

- Define the oscillation norm of the Hamiltonian to be

$$\|H_s\|_{osc} = \int_0^1 \left( \max_{x \in P \times \mathbb{R}} H_s - \min_{x \in P \times \mathbb{R}} H_s \right) ds.$$ 

- Following Dimitroglou Rizell-Sullivan, denote by $l(\gamma)$ the length of a Reeb chord $\gamma$, and let

$$c_i(\Lambda) = \min \{ l(\gamma) | \gamma \text{ is a Reeb chord, } \deg(\gamma) = i \text{ or } n - i \}. $$

Order them so that $c_{j_0}(\Lambda) \geq c_{j_1}(\Lambda) \geq ... \geq c_{j_n}(\Lambda)$. 
**Theorem (L.)**

Let $M$ be orientable, $\Lambda \subset J^1(M)$ be a closed Legendrian submanifold, and $k$ be a field ($\Lambda$ is spin if $\text{char} \, k \neq 2$). Suppose there exists a $k$-coefficient sheaf $\mathcal{F} \in \mathcal{Sh}_\Lambda^b(M \times \mathbb{R})$ with microlocal rank $r$ such that $\text{supp}(\mathcal{F})$ is compact. Let $H_s$ be any compactly supported Hamiltonian such that for some $0 \leq k \leq n$

$$\|H_s\|_{\text{osc}} < c_{jk}(\Lambda)$$

and $\varphi^1_H(\Lambda)$ is transverse to the Reeb flow applied to $\Lambda$. Then the number of Reeb chords between $\Lambda$ and $\varphi^1_H(\Lambda)$ is

$$Q(\Lambda, \varphi^1_H(\Lambda)) \geq \sum_{i=0}^{k} b_j(\Lambda; \mathbb{R}; k).$$
Visualizing Reeb Chords

- We study the morphism between two sheaves $\mathcal{F}, \mathcal{G} \in Sh^b_\Lambda(M \times \mathbb{R}_t)$ by fixing $\mathcal{F}$ and considering positive/negative Reeb pushoffs of $\mathcal{G}$.

Following Tamarkin (and Guillermou, Schapira, Shende etc.), denote the movie of $\mathcal{F}$ under the identity flow by $\mathcal{F}_q$. Denote the movie of $\mathcal{G}$ under the Reeb flow by $\mathcal{G}_r$ (in $M \times \mathbb{R}_t \times \mathbb{R}_u$). Consider $\text{Hom}(\mathcal{F}_q, \mathcal{G}_r)$.
We study the morphism between two sheaves $\mathcal{F}, \mathcal{G} \in Sh^b_{\Lambda}(M \times \mathbb{R}_t)$ by fixing $\mathcal{F}$ and considering positive/negative Reeb pushoffs of $\mathcal{G}$.

Following Tamarkin (and Guillermou, Schapira, Shende etc.), denote the movie of $\mathcal{F}$ under the identity flow by $\mathcal{F}_q$. Denote the movie of $\mathcal{G}$ under the Reeb flow by $\mathcal{G}_r$ (in $M \times \mathbb{R}_t \times \mathbb{R}_u$). Consider $\mathcal{H}om(\mathcal{F}_q, \mathcal{G}_r)$. 

![Diagram of Reeb chords and sheaves](image-url)
Asano-Ike 17’ defined a persistence distance for sheaves on $M \times \mathbb{R}_t$, and showed that the distance is bounded by the oscillation norm $\|H\|_{\text{osc}}$. 

Define $\text{Hom}(\mathbb{R}, \mathbb{R}) (F, G) = u^* \text{Hom}(F_q, G_r)$. We show that the distance of Asano-Ike descends to $\mathbb{R}_u$ and measures how fast the intervals on $\mathbb{R}_u$ vary.
Asano-Ike 17’ defined a persistence distance for sheaves on $M \times \mathbb{R}_t$, and showed that the distance is bounded by the oscillation norm $\| H \|_{osc}$.

Define $\mathcal{H}om_{(-\infty, +\infty)}(\mathcal{F}, \mathcal{G}) = u_* \mathcal{H}om(\mathcal{F}_q, \mathcal{G}_r)$. We show that the distance of Asano-Ike descends to $\mathbb{R}_u$ and measures how fast the intervals on $\mathbb{R}_u$ vary.
Define $\text{Hom}_-(\mathcal{F}, \mathcal{G}) = \Gamma(u^{-1}([-\epsilon, +\infty)), \mathcal{H}om(\mathcal{F}_q, \mathcal{G}_r))$ and $\text{Hom}_+(\mathcal{F}, \mathcal{G}) = \Gamma(u^{-1}([\epsilon, +\infty)), \mathcal{H}om(\mathcal{F}_q, \mathcal{G}_r))$.

**Theorem (L.)**

For $\mathcal{F}, \mathcal{G} \in \text{Sh}_\Lambda^b(M \times \mathbb{R})$ where $\dim M = n$, suppose $\text{supp}(\mathcal{F})$ and $\text{supp}(\mathcal{F})$ are compact. Then

1. $\text{Hom}_-(\mathcal{F}, \mathcal{G}) \simeq \text{Hom}(\mathcal{F}, \mathcal{G}_{-\epsilon}) \simeq \mathcal{F}^\vee \otimes^L \mathcal{G}$, and $\text{Hom}_+(\mathcal{F}, \mathcal{G}) \simeq \text{Hom}(\mathcal{F}, \mathcal{G}_\epsilon) \simeq \text{Hom}(\mathcal{F}, \mathcal{G})$;

2. if $M$ is orientable, then there is a duality $\text{Hom}_+(\mathcal{F}, \mathcal{G}) \simeq \text{Hom}_-(\mathcal{F}, \mathcal{G})^\vee \{-n - 1\}$;

3. if $\mathcal{F}$ has microstalk $F$, then there is an exact triangle

$$\text{Hom}_-(\mathcal{F}, \mathcal{F}) \to \text{Hom}_+(\mathcal{F}, \mathcal{F}) \to C^*(\Lambda; \text{Hom}(F, F)) \to .$$
Conjecture 1 on relative Calabi-Yau

- The duality exact sequence also holds for different sheaves $\mathcal{F}$ and $\mathcal{G}$ (though the third term may be replaced by cochains on $\Lambda$ twisted by a local system). In fact we conjecture that the duality and exact sequence fit into a commutative diagram.
Conjecture 1 on relative Calabi-Yau

- The duality exact sequence also holds for different sheaves $\mathcal{F}$ and $\mathcal{G}$ (though the third term may be replaced by cochains on $\Lambda$ twisted by a local system). In fact we conjecture that the duality and exact sequence fit into a commutative diagram.

- Suppose $\mathcal{S}_{\Lambda,+}^b(M \times \mathbb{R})_0$ (resp. $\mathcal{S}_{\Lambda,-}^b(M \times \mathbb{R})_0$) be the subcategory consisting only of sheaves with compact support with morphisms being $\text{Hom}_+(-,-)$ (resp. $\text{Hom}_-(-,-)$). Then

$$
\begin{array}{cccc}
\mathcal{S}_{\Lambda,+}^b(M \times \mathbb{R})_0[n] & \xrightarrow{m_\Lambda[n]} & m_\Lambda^* \text{Loc}^b(\Lambda)[n] & \longrightarrow & \mathcal{S}_{\Lambda,-}^b(M \times \mathbb{R})_0[n+1] \\
\downarrow & & \downarrow PD & & \downarrow \\
\mathcal{S}_{\Lambda,-}^b(M \times \mathbb{R})_0^\vee[-1] & \longrightarrow & (m_\Lambda^* \text{Loc}^b(\Lambda))^\vee & \xrightarrow{m_\Lambda^\vee} & \mathcal{S}_{\Lambda,+}^b(M \times \mathbb{R})_0^\vee,
\end{array}
$$

which should suggest that $m_\Lambda : \mathcal{S}_{\Lambda,+}^b(M \times \mathbb{R})_0 \rightarrow \text{Loc}^b(\Lambda)$ is a relative right Calabi-Yau functor.
Conjecture 2 on filtered augmentations and sheaves

- Dimitroglou Rizell-Sullivan assumed the existence of an augmentation of the sub-algebra $LCH_\ast(\Lambda)$ generated by Reeb chords shorter than $l$ and get the estimation on chords.
Conjecture 2 on filtered augmentations and sheaves

- Dimitroglou Rizell-Sullivan assumed the existence of an augmentation of the sub-algebra $LCH^*_\Lambda$ generated by Reeb chords shorter than $l$ and get the estimation on chords.
- We conjecture that by assuming existence of a single sheaf $\mathcal{F} \in Sh_{\Lambda_q \cup \Lambda_r}^b(M \times \mathbb{R}_t \times [0, l))$ one can get the same estimate.
Conjecture 2 on filtered augmentations and sheaves

- Dimitroglou Rizell-Sullivan assumed the existence of an augmentation of the sub-algebra $LCH_\ast^!(\Lambda)$ generated by Reeb chords shorter than $l$ and get the estimation on chords.

- We conjecture that by assuming existence of a single sheaf $F \in \mathcal{Sh}^b_{\Lambda \cup \Lambda}(M \times \mathbb{R}_t \times [0, l))$ one can get the same estimate.

- When $l$ is less than the shortest Reeb chords of $\Lambda$, such an augmentation always exists (easy) and so does such a sheaf (a deep theorem of Guillermou 12'). The estimation is then done in Asano-Ike 20' (they deal with a more general case of immersed Lagrangians that only lift to $T^*M \times S^1$).