Estimating Reeb Chords Using Microlocal Sheaf Theory

Symplectic Zoominar

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Contact Manifolds and Legendrians

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- The Reeb vector field of $(J^1(M), \alpha_{\mathrm{std}})$ is $R_\alpha = \partial_t$. Hence Reeb chords on Λ correspond to the immersed points in the Lagrangian projection.

Microlocal Sheaves

- For a sheaf $\mathscr{F}$ on X, its singular support $SS^\infty(\mathscr{F}) \subset T^{*,\infty}X$ encodes "the points and codirections where the stalk of the sheaf jumps". It is always Legendrian or coisotropic (Kashiwara-Schapira).

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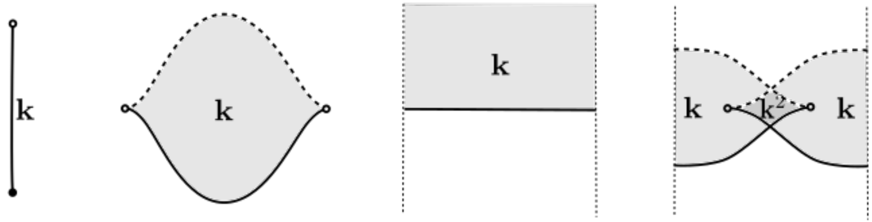
Theorem (Guillermou-Kashiwara-Schapira 12')

The category $Sh^b_\Lambda(X)$ of sheaves with $SS^\infty(\mathscr{F}) \subset \Lambda$ is invariant under Legendrian isotopies.

Microlocal Sheaves

- Viewing $\Lambda \subset J^1(M)$ as in $T^{*,\infty}(M \times \mathbb{R})$ (via $(x, \xi, t) \mapsto (x, \xi, t, 1)$), one can consider sheaves with singular support on Λ.



$M = \mathrm{pt}$ $M = \mathbb{R}$ $M = S^1$ $M = S^1$

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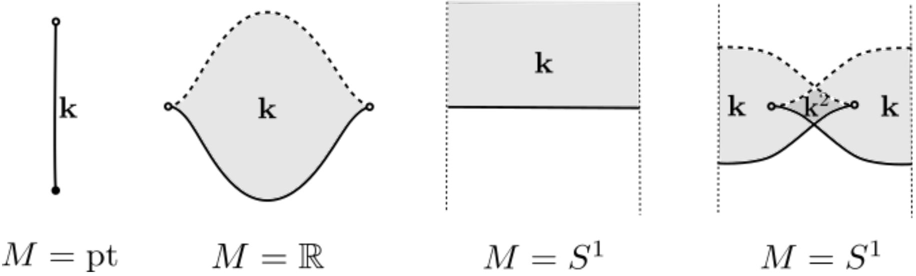
- Sheaves in $Sh^b_\Lambda(M \times \mathbb{R})$ are sheaves that are locally constant on each smooth stratum of the front $\pi(\Lambda)$ (plus extra conditions).



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- For self Reeb chords of Λ, the Arnol'd type bound one would expect in good situations should be $\frac{1}{2}\sum_{i=0}^{n} b_i(\Lambda)$.
- Let $\mathcal{Q}(\Lambda)$ (resp. $\mathcal{Q}_i(\Lambda)$) be the set of all (resp. degree i) Reeb chords on Λ. Suppose $\pi_{\mathrm{Lag}}(\Lambda)$ is immersed with transverse double points.

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Theorem (L.)

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1. *If there exists a $\mathbb{k}$-coefficient sheaf $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ with microlocal rank r such that $\mathrm{supp}(\mathscr{F})$ is compact, then*

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2. *If there exists a $\mathbb{k}$-coefficient sheaf $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ with perfect (micro)stalk such that $\mathrm{supp}(\mathscr{F})$ is compact, then*
$$|\mathcal{Q}(\Lambda)| \geq \frac{1}{2}\sum_{i=0}^{n} b_i(\Lambda; \mathbb{k}).$$

Remarks on Main Result I

- Conjecturally, microlocal rank r sheaves are equivalent to r-dimensional representations of the Legendrian contact homology (when $r = 1$ they are called augmentations).

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- We can show that the existence of a sheaf plus **horizontal displaceability** implies **compact support** of the sheaf, but the converse is false. On the other hand, existence of generating families linear at infinity probably implies existence of sheaves over any ring, but there are Legendrians that only admit sheaves over certain rings.

Main Result II

- Consider the Reeb chords between Λ and $\varphi_H^1(\Lambda)$. When H can be lifted from a symplectic Hamiltonian H_{symp}, this is just the Lagrangian intersection between $\pi_{\mathrm{Lag}}(\Lambda)$ and $\varphi^1_{H_{\mathrm{symp}}}(\pi_{\mathrm{Lag}}(\Lambda))$.

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- Define the oscillation norm of the Hamiltonian to be

$$\|H_s\|_{\mathrm{osc}} = \int_0^1 \left(\max_{x \in P \times \mathbb{R}} H_s - \min_{x \in P \times \mathbb{R}} H_s \right) ds.$$

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- Following Dimitroglou Rizell-Sullivan, denote by $l(\gamma)$ the length of a Reeb chord γ, and let

$$c_i(\Lambda) = \min\{l(\gamma) | \gamma \text{ is a Reeb chord, } \deg(\gamma) = i \text{ or } n - i\}.$$

Order them so that $c_{j_0}(\Lambda) \geq c_{j_1}(\Lambda) \geq ... \geq c_{j_n}(\Lambda)$.

Main Result II

Theorem (L.)

Let M be orientable, $\Lambda \subset J^1(M)$ be a closed Legendrian submanifold, and $\Bbbk$ be a field (Λ is spin if $\mathrm{char}\,\Bbbk \neq 2$). Suppose there exists a $\Bbbk$-coefficient sheaf $\mathscr{F} \in Sh^b_\Lambda(M \times \mathbb{R})$ with microlocal rank r such that $\mathrm{supp}(\mathscr{F})$ is compact. Let H_s be any compactly supported Hamiltonian such that for some $0 \leq k \leq n$

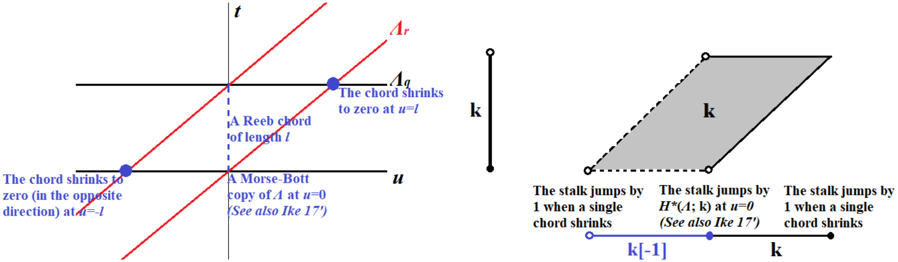
$$\|H_s\|_{osc} < c_{j_k}(\Lambda)$$

and $\varphi_H^1(\Lambda)$ is transverse to the Reeb flow applied to Λ. Then the number of Reeb chords between Λ and $\varphi_H^1(\Lambda)$ is

$$\mathcal{Q}(\Lambda, \varphi_H^1(\Lambda)) \geq \sum_{i=0}^{k} b_{j_i}(\Lambda; \Bbbk).$$

Visualizing Reeb Chords

- We study the morphism between two sheaves $\mathscr{F}, \mathscr{G} \in Sh^b_\Lambda(M \times \mathbb{R}_t)$ by fixing $\mathscr{F}$ and considering positive/negative Reeb pushoffs of $\mathscr{G}$.



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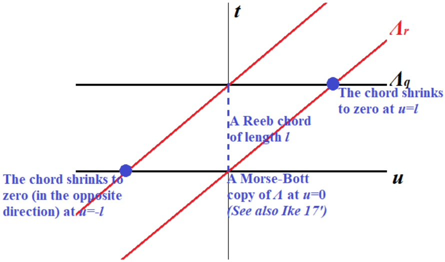
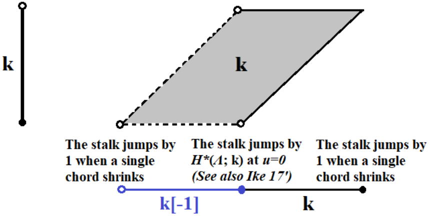
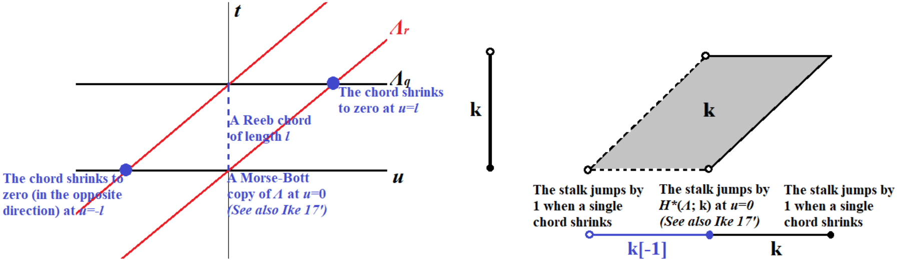
- Following Tamarkin (and Guillermou, Schapira, Shende etc.), denote the movie of $\mathscr{F}$ under the identity flow by $\mathscr{F}_q$. Denote the movie of $\mathscr{G}$ under the Reeb flow by $\mathscr{G}_r$ (in $M \times \mathbb{R}_t \times \mathbb{R}_u$). Consider $\mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r)$.





Persistence Structure

- Asano-Ike 17' defined a persistence distance for sheaves on $M \times \mathbb{R}_t$, and showed that the distance is bounded by the oscillation norm $\|H\|_{\mathrm{osc}}$.



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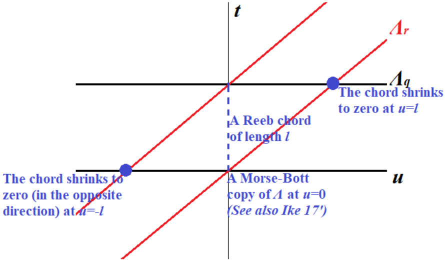
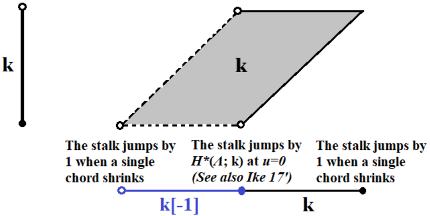
- Define $\mathscr{H}om_{(-\infty,+\infty)}(\mathscr{F},\mathscr{G}) = u_*\mathscr{H}om(\mathscr{F}_q,\mathscr{G}_r)$. We show that the distance of Asano-Ike descends to $\mathbb{R}_u$ and measures how fast the intervals on $\mathbb{R}_u$ vary.





Duality Exact Triangle

- Define $Hom_-(\mathscr{F}, \mathscr{G}) = \Gamma(u^{-1}([-\epsilon, +\infty)), \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r))$ and $Hom_+(\mathscr{F}, \mathscr{G}) = \Gamma(u^{-1}([\epsilon, +\infty)), \mathscr{H}om(\mathscr{F}_q, \mathscr{G}_r))$.

Theorem (L.)

For $\mathscr{F}, \mathscr{G} \in Sh^b_\Lambda(M \times \mathbb{R})$ where $\dim M = n$, suppose $\operatorname{supp}(\mathscr{F})$ and $\operatorname{supp}(\mathscr{F})$ are compact. Then

1. $Hom_-(\mathscr{F}, \mathscr{G}) \simeq Hom(\mathscr{F}, \mathscr{G}_{-\epsilon}) \simeq \mathscr{F}^\vee \otimes^L \mathscr{G}$, *and* $Hom_+(\mathscr{F}, \mathscr{G}) \simeq Hom(\mathscr{F}, \mathscr{G}_\epsilon) \simeq Hom(\mathscr{F}, \mathscr{G})$;
2. *if M is orientable, then there is a duality* $Hom_+(\mathscr{F}, \mathscr{G}) \simeq Hom_-(\mathscr{F}, \mathscr{G})^\vee[-n-1]$;
3. *if $\mathscr{F}$ has microstalk F, then there is an exact triangle* $Hom_-(\mathscr{F}, \mathscr{F}) \to Hom_+(\mathscr{F}, \mathscr{F}) \to C^*(\Lambda; Hom(F, F)) \xrightarrow{+1}$.

Conjecture 1 on relative Calabi-Yau

- The duality exact sequence also holds for different sheaves $\mathscr{F}$ and $\mathscr{G}$ (though the third term may be replaced by cochains on Λ twisted by a local system). In fact we conjecture that the duality and exact sequence fit into a commutative diagram.

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- Suppose $Sh^b_{\Lambda,+}(M \times \mathbb{R})_0$ (resp. $Sh^b_{\Lambda,-}(M \times \mathbb{R})_0$) be the subcategory consisting only of sheaves with compact support with morphisms being $Hom_+(-,-)$ (resp. $Hom_-(-,-)$). Then

$$\begin{tikzcd}
Sh^b_{\Lambda,+}(M \times \mathbb{R})_0[n] \arrow[r, "m_\Lambda{[n]}"] \arrow[d] & m^*_\Lambda Loc^b(\Lambda)[n] \arrow[r] \arrow[d, "PD"] & Sh^b_{\Lambda,-}(M \times \mathbb{R})_0[n+1] \arrow[d] \\
Sh^b_{\Lambda,-}(M \times \mathbb{R})_0^\vee[-1] \arrow[r] & (m^*_\Lambda Loc^b(\Lambda))^\vee \arrow[r, "m^\vee_\Lambda"] & Sh^b_{\Lambda,+}(M \times \mathbb{R})_0^\vee,
\end{tikzcd}$$

which should suggest that $m_\Lambda : Sh^b_{\Lambda,+}(M \times \mathbb{R})_0 \to Loc^b(\Lambda)$ is a relative right Calabi-Yau functor.

Conjecture 2 on filtered augmentations and sheaves

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- When l is less than the shortest Reeb chords of Λ, such an augmentation always exists (easy) and so does such a sheaf (a deep theorem of Guillermou 12'). The estimation is then done in Asano-Ike 20' (they deal with a more general case of immersed Lagrangians that only lift to $T^*M \times S^1$).