Simplicial descent for Chekanov–Eliashberg dg-algebras

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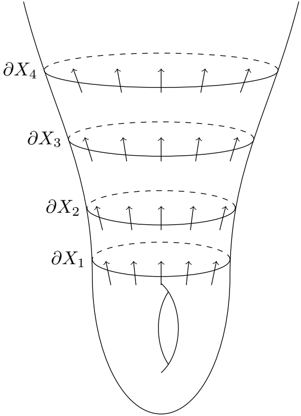
Context and motivation

Definition

A *Weinstein manifold* is an exact symplectic manifold $(X^{2n}, \omega = d\lambda)$ such that:

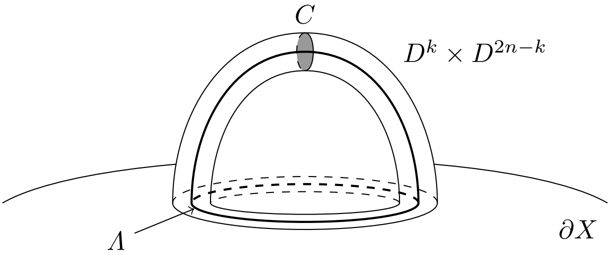
1. The Liouville vector field Z defined by $\omega(Z, -) = \lambda$ is complete.
2. There exists an exhaustion $X = \bigcup_{k=1}^{\infty} X_k$ by compact domains $X_k \subset X$ with smooth boundaries such that Z points outwards along ∂X_k.
3. There exists an exhausting (generalized) Morse function $\phi\colon X \longrightarrow \mathbb{R}$ constant along ∂X_k, such that Z is gradient-like for ϕ.



Weinstein handle of index $0 \leq k \leq n$

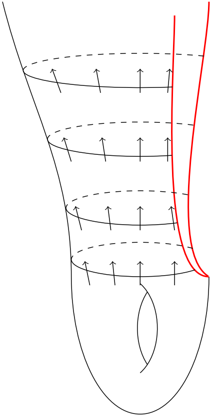
$$\left(D^k \times D^{2n-k}, \sum_{j=1}^{k} (2x_j dy_j + y_j dx_j) + \frac{1}{2} \sum_{j=k+1}^{n} (x_j dy_j - y_j dx_j) \right)$$

- Core disk $L = D^k \times \{\mathbf{0}\}$, attaching sphere $\Lambda = \partial L$
- Cocore disk $C = \{\mathbf{0}\} \times D^{2n-k}$



Definition

A *Weinstein sector* is a Weinstein manifold-with-boundary X^{2n} such that there exists a smooth function $I\colon \partial X \longrightarrow \mathbb{R}$ that is linear at infinity and whose Hamiltonian vector field X_I points outwards along ∂X.



Consequence

Near ∂X there are coordinates of the form $(V^{2n-2} \times T^*(-\delta, 0], \lambda_V + pdq)$ where V is called the *symplectic boundary* of X.

Example

If M is a manifold-with-boundary then $(T^*M, \lambda = pdq)$ is a Weinstein sector. Symplectic boundary is $T^*(\partial M)$.

Definition

Let X be a Weinstein manifold. A *sectorial cover* is a cover $X = X_1 \cup \cdots \cup X_m$ where X_i is a Weinstein manifold-with-boundary such that there are functions $I_i \colon \partial X_i \longrightarrow \mathbb{R}$ (linear at infinity) such that

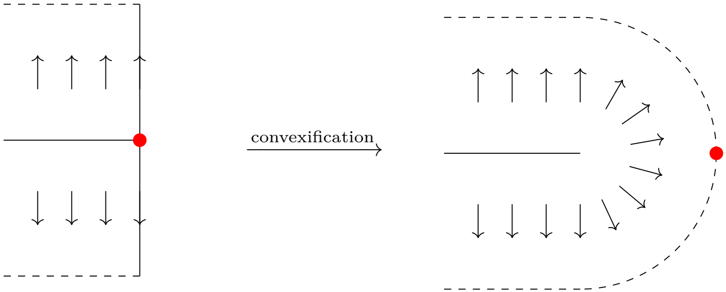
1. X_{I_i} points outwards along ∂X_i
2. X_{I_i} is tangent to ∂X_j for $i \neq j$
3. $[X_{I_i}, X_{I_j}] = 0$

Consequence

Near $\bigcap_{i \in I} \partial X_i$ there are coordinates of the form

$$V_I^{2n-2k} \times T^*(-\delta, \delta)^k$$

where $k = |I|$.



Weinstein sector X with symplectic boundary V

Weinstein pair (X', V)

Fact (Ganatra–Pardon–Shende)

Let $X = X_1 \cup \cdots \cup X_m$ be a sectorial cover. Then there is a pre-triangulated equivalence of A_∞-categories

$$\mathcal{W}(X) \cong \underset{\varnothing \neq I \subset \{1,\dots,m\}}{\mathrm{hocolim}} \mathcal{W}\left(\bigcap_{i \in I} X_i\right)$$

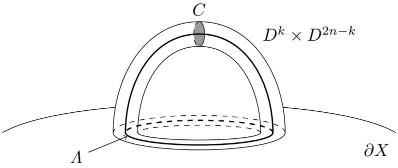
Fact (Chantraine–Dimitroglou Rizell–Ghiggini–Golovko, GPS)

$\mathcal{W}(X)$ is generated by the cocore disks C of the critical Weinstein handles.

Fact (Bourgeois–Ekholm–Eliashberg)

There is an A_∞-quasi-isomorphism

$$CW^*(C) \cong CE^*(\Lambda)$$

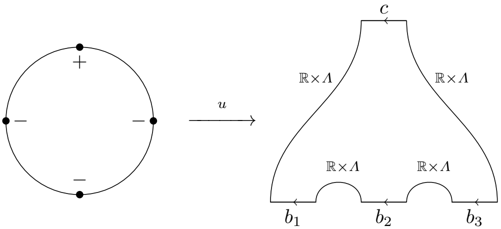


Question:

Does a sectorial cover give rise to a local-to-global principle for CE^*(attaching spheres)?

Chekanov–Eliashberg dg-algebra $CE^*(\Lambda)$

- Generators: Reeb chords of $\Lambda \subset \partial X$
- Grading: Conley–Zehnder index
- Differential: Counts rigid J-holomorphic disks in $\mathbb{R} \times \partial X$ with boundary on $\mathbb{R} \times \Lambda$



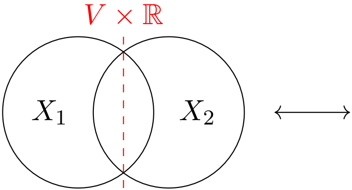
Simplicial decompositions

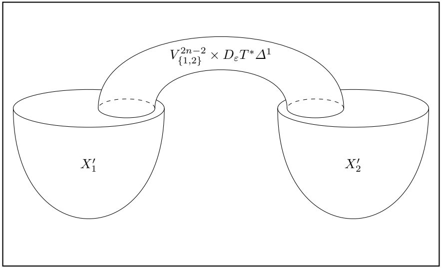
Definition

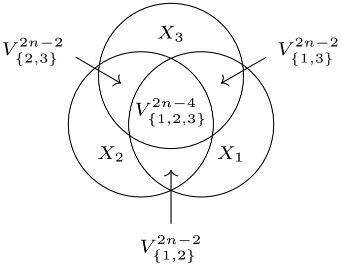
A sectorial cover $X = X_1 \cup \cdots \cup X_m$ is *good* if for any $\varnothing \neq A \subset \{1, \ldots, m\}$ such that $\bigcap_{i \in A} X_i \neq \varnothing$ we have

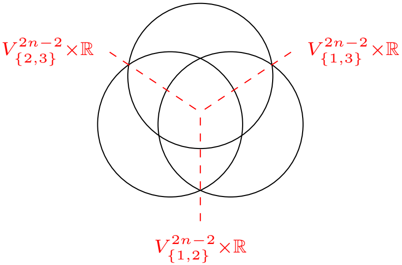
$$N\left(\bigcap_{i \in A} X_i\right) \cong V_A^{2n-2k} \times T^*\mathbb{R}^k$$

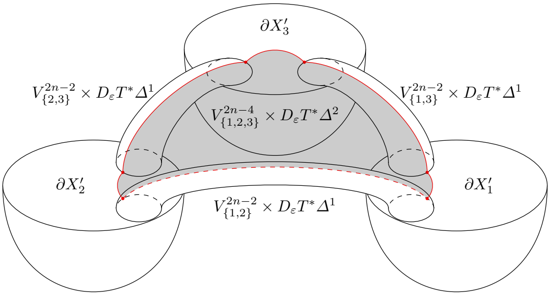
where $k = |A| - 1$.






X_3
$V^{2n-2}_{\{2,3\}}$
$V^{2n-2}_{\{1,3\}}$
$V^{2n-4}_{\{1,2,3\}}$
X_2
X_1
$V^{2n-2}_{\{1,2\}}$


$V^{2n-2}_{\{2,3\}} \times \mathbb{R}$
$V^{2n-2}_{\{1,3\}} \times \mathbb{R}$
$V^{2n-2}_{\{1,2\}} \times \mathbb{R}$

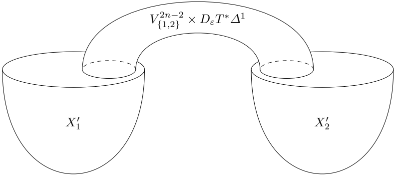

$\partial X_3'$
$V^{2n-2}_{\{2,3\}} \times D_\varepsilon T^* \Delta^1$
$V^{2n-2}_{\{1,3\}} \times D_\varepsilon T^* \Delta^1$
$V^{2n-4}_{\{1,2,3\}} \times D_\varepsilon T^* \Delta^2$
$\partial X_2'$
$\partial X_1'$
$V^{2n-2}_{\{1,2\}} \times D_\varepsilon T^* \Delta^1$

Definition

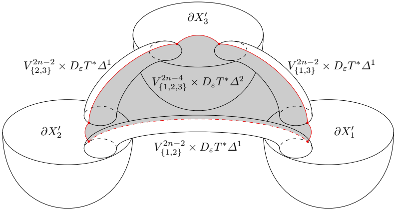
A *simplicial decomposition* of X is a triple $(C, \boldsymbol{V}, \boldsymbol{A})$ where

- C is a simplicial complex
- $\boldsymbol{V}$ *handle data*
- $\boldsymbol{A}$ *attaching data*

} Sets containing Weinstein manifolds and Weinstein hypersurfaces



$C = \Delta^1$



$C = \Delta^2$

Theorem (A.)

Let $\Sigma(\boldsymbol{h})$ be the union of Legendrian attaching spheres of X adapted to $(C, \boldsymbol{V}, \boldsymbol{A})$. Then there is an isomorphism of dg-algebras

$$CE^*(\Sigma(\boldsymbol{h}); X_0) \cong \operatorname*{colim}_{\sigma_k \in C_k} \mathcal{A}_{\sigma_k}$$

Diagram

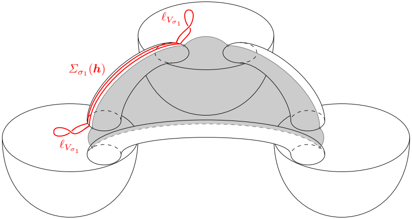
- Associated to each $\sigma_k \in C_k$ there is a dg-algebra $\mathcal{A}_{\sigma_k}$
- For each $\sigma_k \subset \sigma_{k+1}$ we have $\mathcal{A}_{\sigma_{k+1}} \subset \mathcal{A}_{\sigma_k}$

Legendrian attaching data and dg-subalgebras

- Let $\boldsymbol{h}$ be the collection of handle decompositions of all Weinstein manifolds $V \in \boldsymbol{V} \cup \boldsymbol{A}$.
- Remove all the critical Weinstein handles of X
- Let $\Sigma(\boldsymbol{h})$ be the union of the Legendrian attaching spheres of the critical Weinstein handles of X.

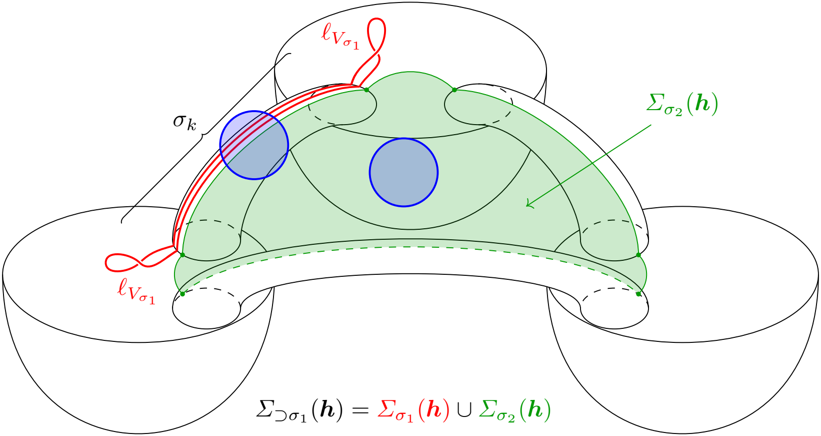
$$\Sigma(\boldsymbol{h}) = \bigcup_{\substack{\sigma_k \in C_k \\ 0 \le k \le m}} \Sigma_{\sigma_k}(\boldsymbol{h})$$

$$\Sigma_{\supset \sigma_k}(\boldsymbol{h}) = \bigcup_{\substack{\sigma_i \supset \sigma_k \\ k \le i \le m}} \Sigma_{\sigma_i}(\boldsymbol{h})$$



- Reeb chords of $\Sigma(\boldsymbol{h})$ are located in the "center" of each handle corresponding to each $\sigma_k \in C_k$

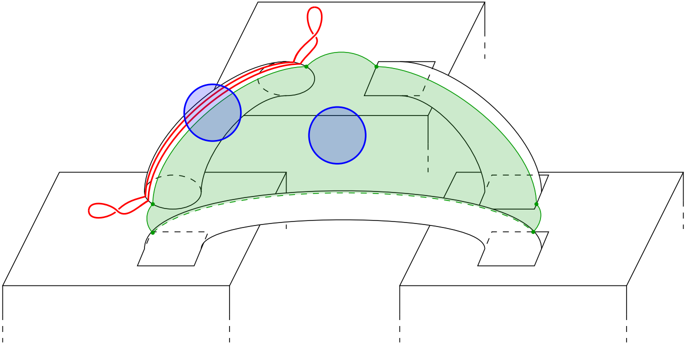
$\mathcal{A}_{\sigma_k}$ is generated by Reeb chords of $\Sigma_{\supset\sigma_k}(\boldsymbol{h})$ located in parts of ∂X_0 corresponding to σ_i for $\sigma_i \supset \sigma_k$.



In fact we have

$$\mathcal{A}_{\sigma_k} \cong CE^*(\Sigma_{\supset \sigma_k}(\boldsymbol{h}); X(\sigma_k)_0)$$

$X(\sigma_k)$ is obtained by replacing $V_{\sigma_i}^{2n-2i} \in \boldsymbol{V} \cup \boldsymbol{A}$ with a half symplectization of a contactization for every $\sigma_i \not\supset \sigma_k$.



Theorem (A.)

Let $\Sigma(\boldsymbol{h})$ be the union of Legendrian attaching spheres of X adapted to $(C, \boldsymbol{V}, \boldsymbol{A})$. Then there is an isomorphism of dg-algebras

$$CE^*(\Sigma(\boldsymbol{h}); X_0) \cong \operatorname*{colim}_{\sigma_k \in C_k} CE^*(\Sigma_{\supset \sigma_k}(\boldsymbol{h}); X(\sigma_k)_0)$$

Thank you!