Quantitative Legendrian Geometry

SYMPLECTIC ZOOMINAR

joint with Georgios Dimitroglou Rizell

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inspiration	Symplectic	Lagrangian
	Contact	Legendrian

Me

(1) ROW 1 in the 80s

Non-squeezing (Gromov '85) If $B^{2n}_R \xrightarrow[\text{embed}]{\text{symplect}} Z^{2n}_r$

then $R \leq r$

$\{z \in \mathbb{R}^{2n} \mid \|z\| < R\}$ $\{z \in \mathbb{R}^{2n} \mid x_1^2 + y_1^2 < r^2\}$

C^0 (Eliashberg–Gromov?) $\phi_k \in \mathrm{Symp}(W^{2n}, \omega)$

$\phi_k \xrightarrow{C^0} \phi_\infty$, ϕ_∞ diffeo then $\phi_\infty \in \mathrm{Symp}(W, \omega)$

Displacement (Floer '88 + Oh + ...) If $L^n \subset (W^{2n}, \omega)$

Lagrangian ($\omega|_{TL} = 0$) & $\phi_t \in \mathrm{Ham}(W)$ $\left(\frac{d\phi_t}{dt} = X_H \circ \phi_t \quad \omega(X_H, \cdot) = dH \text{ for some Ham } H_t : W \to \mathbb{R} \right)$

& $L \pitchfork \phi_1(L)$ then

$$\# \, L \cap \phi_1(L) \geq \sum_{i=0}^{n} \dim H_i(L)$$

(not true in general)

$\boxed{\text{ROW 1 in the} \geq 90\text{'s}}$

C^0 (Laudenbach–Sikorav '94, Opshtein '09, Humilière–Leclercq–Seyfaddini '15)

If $\phi_k \in \mathrm{Symp}(W)$, $\phi_k \xrightarrow{C^0} \phi_\infty$, ϕ_k homeo, $C \subset W$ coisotropic submfd $(TC^\perp \subset TC)$,
and $\phi_\infty(C)$ smooth, then
$\phi_\infty(C)$ coisotropic

Given $\phi \in Ham(W)$ define Hofer norm

$$\|\phi\| = \inf\left\{ \|H\| := \int_0^1 \left(\max_W H(\cdot,t) - \min_W H(\cdot,t)\right)dt \;\middle|\; \phi = \phi_H^1 \right\}$$

Define Hofer metric on $Ham(W)$ by

$$\delta(\phi,\psi) = \|\phi^{-1}\circ\psi\|$$

For $L^n \subset W^{2n}$ Lagrangian define orbitspace

$$\mathcal{L}(L) = \{ L' \mid L' = \phi(L) \text{ for } \phi \in Ham(W) \}$$

Define Chekanov-Hofer metric on $\mathcal{L}(L)$ by

$$\delta(L_0,L_1) = \left\{ \inf_{\phi \in Ham(W)} \|\phi\| \;\middle|\; \phi(L_0) = L_1 \right\}$$

METRIC

(Lalonde-McDuff '95) Hofer metric nondegenerate

(Chekanov '00) Chekanov-Hofer metric nondegenerate

Displacement (Chekanov '00) LCW Lag. If

$\|H\| <$ min area of J holo disk / sphere & $L \pitchfork \phi^1_H(L)$

then $\# \phi^1_H(L) \cap L \geq \sum_{i=0}^{n} \dim H_i(L)$

③ FLOER RECIPE

(1) Put in geometric data (ω, L, φ_H, etc.) (symp, Lag, Ham). Stir in almost complex structure J.

(2) Hypothesize/prove moduli space of J-holo curves $\mathcal{M}$ is compact á la Gromov. Analysis $\Rightarrow$ $\mathcal{M}$ mfd.

(3) Use $\mathcal{M}^0, \mathcal{M}^1, \mathcal{M}^{-1}$ to define Floer chain complex CF_*(data). Extract useful algebraic info like singular homology, barcode, etc,

(4) Use (limited?) invariance of algebraic info to serve desired rigid quantitative result.

④ ATTEMPTS TO TRANSLATE ROW 1 TO ROW 2

symplectic / lag — contact / leg

Translate Step 1 (Geometric Data)

$L^n \subset (W^{2n}, \omega) \longrightarrow \Lambda^n \subset (M^{2n+1}, \xi = \ker(\alpha))$

contact mfld

Legendrian $\Lambda \subset \xi$

contact distribution

contact 1-form

Hamiltonian $\varphi^t_H \longrightarrow$ Contact Hamiltonian $\varphi^t_{\alpha,H}$ defined by
- $\alpha(X_{\alpha,H}) = H$
- $d\alpha(X_{\alpha,H}, \cdot) = dH(R_\alpha)\alpha - dH$

Reeb vector field R_α defined by

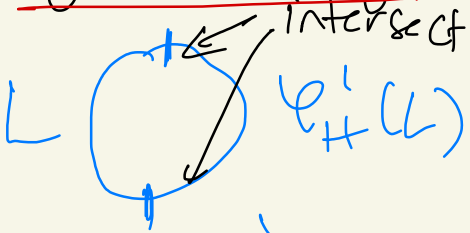
$\alpha(R_\alpha) = 1, \; d\alpha(R_\alpha, \cdot) = 0$

Generators of Floer complex: Lagrangian intersection points $L \cap \varphi^1_H(L)$ $\longrightarrow$ Reeb chords between $\varphi^1_{\alpha,H}(\Lambda)$ & Λ ("mixed") $RC(\Lambda, \varphi^1_{\alpha,H}(\Lambda))$

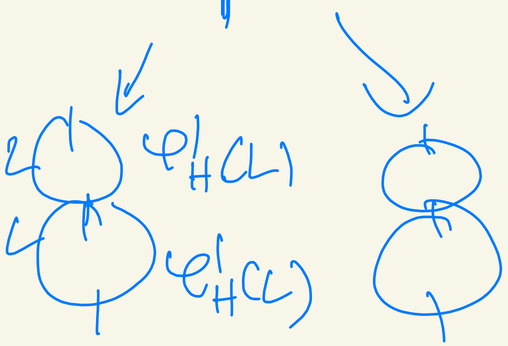
And also Reeb chords from Legendrian to itself ("pure") $RC(\Lambda,\Lambda)$, $RC(\varphi^1_{\alpha,H}(\Lambda), \varphi^1_{\alpha,H}(\Lambda))$

Translation of step 2 (Compactness)

Symplectic geometry



G-cptness

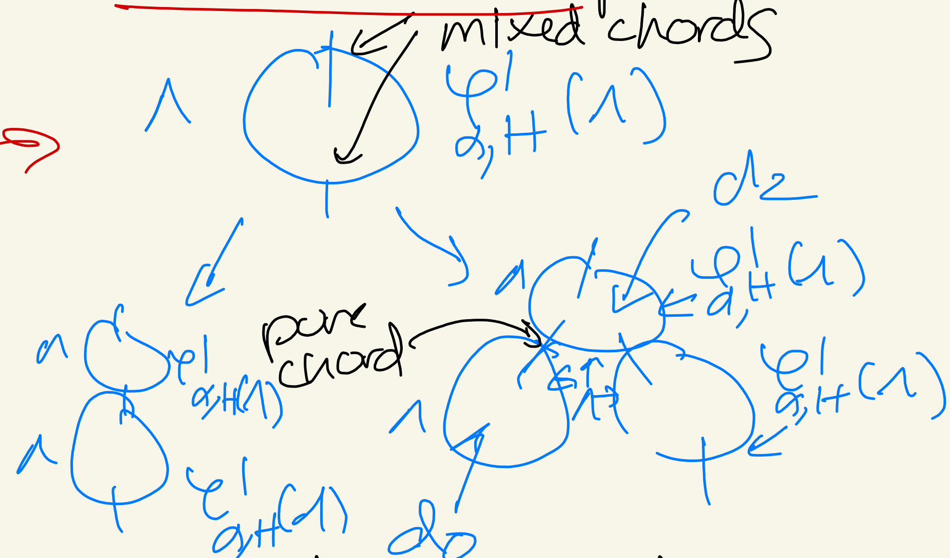


$\rightsquigarrow d^2=0$

$\Rightarrow$ Flor cplx well defined

$CF(L, \varphi^1_H(L))$

Contact Geometry



$\rightsquigarrow d_1 \circ d_1 + d_0 d_2 + d_2 d_0 = 0$

$\Rightarrow$ Chekanov-Eliashberg (CE) differential graded algebra (DGA)

$(A, d) = (A(\Lambda \sqcup \varphi^1_{\alpha,H}(\Lambda)), d)$

well defined

"Legendrian Contact Homology $H_*(A,d)$"

is Leg isotopy invariant.

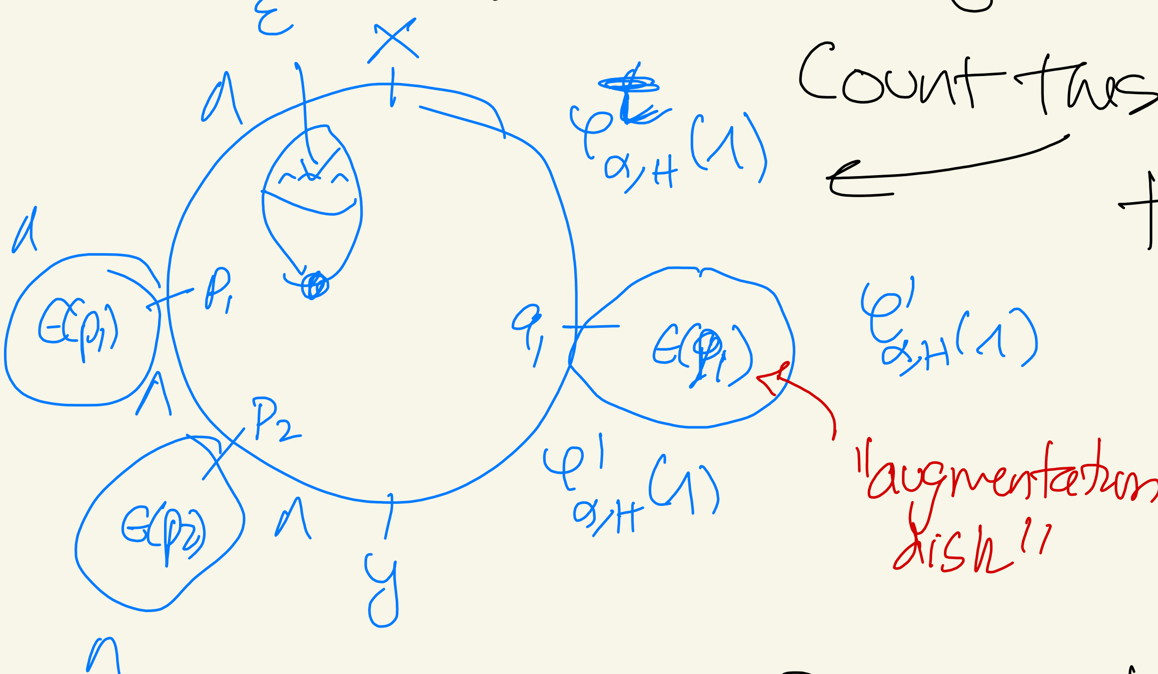
Translation of Step 3 (extract useful algebra info)

IF Λ has an augmentation $\varepsilon : (A, \partial) \xrightarrow{\text{DGA morph}} (\mathbb{R}, 0)$

Ground field/ring — trivial differential

THEN can "linearize" DGA (A, ∂)

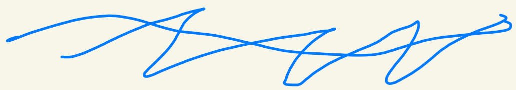
and define (relative) Rabinowitz Floer chain complex generated by mixed chords $RCF(\Lambda, \varphi'_{\alpha,H}(\Lambda))$



Count these "broken disk" to define $\langle dx, y \rangle$ in RCF

Proceed to step 3 as before.

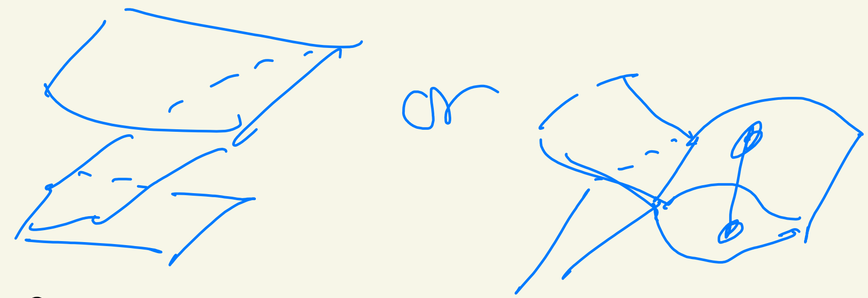
But (Eliashberg (?) / Murphy ('12)) most Λ do not have augmentations.

- 

  C^0 approx knot with highly stabilized knots

- In higher dims, loose Legendrians are C^0-dense in smooth mflds.

And stab/loose $\Rightarrow \exists\ x \in RC(\Lambda)$ s.t. $dx = 1$

$\Rightarrow$ (1) $\epsilon d(x) = \epsilon(1) = 1 \neq 0 = d\epsilon(x) \Rightarrow$ no augmentation

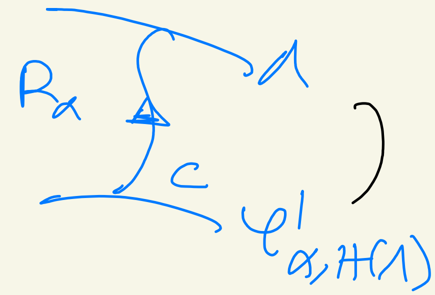
(2) $[1] = [0]$ in $H_*(\mathcal{A}, \partial) \Rightarrow$ Leg Cont Hom invariant trivial

To study Legendrians without augmentations (eg loose) reduce the action window $RCF_{[a,b]}(\Lambda, \varphi^1_{\alpha,H}(\Lambda))$

Details: Define length of Reeb chord $c \in RC$ by

$$\ell(c) = \int c^* \alpha$$

α ← contact form



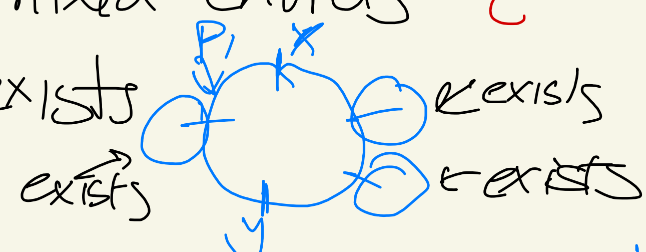
(Too much detail: Length can be 0 if $c \in \Lambda \cap \varphi^t_{\alpha,H}(\Lambda)$ or < 0 for mixed chords)

Stokes thm $\Rightarrow$ ∂ decreases length $\Rightarrow$

$A^\ell :=$ subalg generated by chords of length $< \ell$ is ~~a~~ in fact subDGA.

$\ell(\Lambda) > 0 \Rightarrow$ for all Λ $\exists\, \ell > 0$ s.t. $A^\ell(\Lambda)$ has augmentation

Let $RCF_{[a,b]}(\Lambda, \varphi^1_{\alpha,H}(\Lambda))$ generated by mixed chords c with $a < \ell(c) < b$ s.t. augmentation exists on $A^\ell(\Lambda), A^\ell(\varphi^1_{\alpha,H}(\Lambda))$ with $b - a < \ell$



$0 < \ell(x) - \ell(y) - \ell(p_1) < b - a - \ell(p_1) \Rightarrow \ell(p_1) < \ell \Rightarrow \varepsilon$ exists

(5) TRANSLATING STEP 4: LEGENDRIAN DISPLACEMENT

Recall Chekanov: LCW Lag & $\|H\| <$ min area of disk/sphere $\Rightarrow \#(L \cap \varphi^1_H(L)) \geq \sum \dim H_i(L)$

Thm (Dimitroglou Rizell – '21) Given $\Lambda \subset (M, \xi = \ker(\alpha))$ & contact ham H. Order $RC(\Lambda)$ $0 < \ell(c_1) < \ell(c_2) < \ell(c_3) < \ldots$

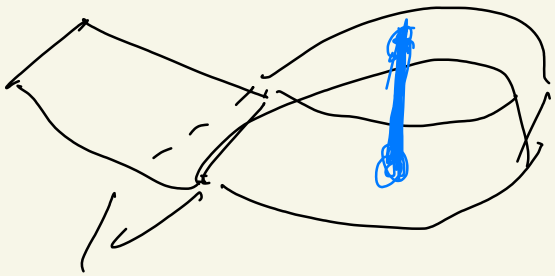
Choose ℓ such that (1) $\mathcal{A}^\ell(\Lambda)$ has aug

(2) If γ is a contractible Reeb orbit & expected dim

$\mathcal{M}(\gamma) \leq 1$ then $\int \gamma^* \alpha > \ell$

If $\|H\| < \min(\ell, \ell(c_k))$

then $\#RC(\Lambda, \varphi^1_{\alpha,H}(\Lambda)) \geq \sum_{i=0}^{n} \dim H_i(\Lambda) - 2k$



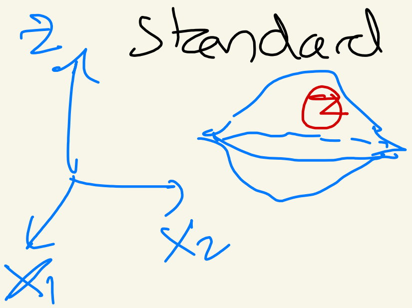
Remarks ① Should be able to improve (2) if polyfold/Kuranishi analysis done for $A(L)$ with coefs in periodic orbit contact homology

② The case $(M, \xi = \ker\alpha) = (P\times\mathbb{R}, dz-\lambda)$ done in (DR- '18) with better lower bounds. Here $(P, d\lambda)$ has finite geometry at ∞.
This case also done in (Li '21) assuming existence of sheaves (similar to assuming existence of representation which generalizes augmentation)

③ (Nakamura '21) If $\dim\Lambda \geq 3$ & Λ has loose chart of size η then displacement energy of Λ is $\leq \eta$.

$\boxed{\text{min } \|H\| \text{ such that } RC(\Lambda, \varphi^1_{\alpha,H}(\Lambda)) = \emptyset}$

(DR – '18) Fix $\delta \ll 1$, let $\Lambda \subset (\mathbb{R}^5, \alpha_{std} = dz - ydx)$ be standard unknot with arb small loose chart with displacement energy $< \delta$.



⑥ TRANSLATING STEP 4 : LEGENDRIAN METRICS

Recall Lalonde-McDuff (resp. Chekanov) : Hofer (resp. Chekanov-Hofer) metric on Ham (resp $\mathcal{L}(L)$) is nondegenerate.

Fix $N \subset (M^{2n+1}, \xi)$ (not necc Legrangian)

Let $\mathcal{L}(N) = \{ N' \mid \varphi(N) = N'$ with $\varphi \in \mathrm{Cont}_0(M, \xi) \}$

Define <u>Chekanov-Hofer-Shelukhin</u> metric on $\mathcal{L}(N)$

$$\delta_\alpha(N, N') = \inf \left\{ \int_0^1 \max |H_t| \, dt \;\middle|\; \Phi^1_{\alpha, H_t}(N) = N' \right\}$$

<u>Def</u> If $\dim N = n$ then N is non-Legendrian if

$\exists x \in N$ s.t. $T_x N \not\subset \xi$

(Rosen - Zhang '18) If N is non deg then $\delta_\alpha \equiv 0$

N is hypertight ~~no con~~ no contractible orbits or $RC(\Lambda) = 0 \in \pi_1(M, \Lambda)$

assume N orderable (does not sit in a pos leg loop)

(Usher '20, Hedicke '21, DR- '21)

If N is leg

then δ_α non deg.

⑦ TRANSLATING STEP 4: LEGENDRIAN C^0-TOPOLOGY

Recall Eliashberg–Gromov, Laudenbach–Sikorav, Opshtein, Humilière–Leclercq–Seyfaddini on C^0 limits of $\varphi_k \in \mathrm{Symp}$ & $\varphi_k(C)$ (C coisotropic)

(Eliasberg '87, Muller–Spaeth '15)

$$\frac{\phi_k \xrightarrow{C^0} \phi_\infty, \ \phi_\infty \text{ diffeo} \Rightarrow \phi_\infty \in \mathrm{Cont}(M, \xi)}{}$$

$\phi_k \in \mathrm{Cont}(M, \xi)$

(DR- '22) Suppose $\varphi_k \in \mathrm{Cont}(M^3, \xi)$, $\phi_k \xrightarrow{C^0} \phi$, ϕ_∞ homeo

ACM Leg, $\phi_\infty(\Lambda)$ smooth. Then ① $\phi_\infty(\Lambda)$ Leg

② $\exists\, \psi \in \mathrm{Cont}(M, \xi)$ s.t. $\psi(\Lambda) = \phi_\infty(\Lambda)$

Prior partial results in higher dimensions

(Rosen-Zhang '18) Prove ① assuming unif conv on

conf fact f_i $(\phi_i^* \alpha = f_i \alpha)$

(Usher '20) Generalized RZ '18 assuming lower bounds on f_i

(Nakamura '20) Prove ① assuming $\exists$ unif lower bound

on $\min\limits_{c \in RC(\phi_i(\Lambda))} \ell(c)$ + tech conditions

DR '21 remove.

6 TRANSLATING STEP 4: LEGENDRIAN SQUEEZING

Recall Gromov's original non squeezing theorem $B^{2n}_R \hookrightarrow Z^{2n}_r \Rightarrow R \leq r$

<u>Def</u> U squeezed onto V if

(Eliashberg-Kim-Polterovich '06)

(PR - '18)

Rmk ① $\partial\Lambda$ has augmentation so 2nd condition necessary

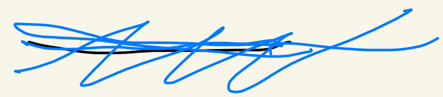
② (Murphy '12) If Λ loose then can be squeezed in $\mathbb{R}$ provided $\exists \Lambda' \subset Y$ such that Λ', Λ have same "formal isotopy data"

③ (Lazarev '19) replaced Λ augmentable to a condition on WFuk. Still not clear what optimal assumption is

From Murphy'12, only "formal data" has a chance of preventing loose leg from squeezing.

For example: bound on # of loose charts does not,
(non 0)

(Ding-Geiges '09) light bulb trick: 

(Eliashberg ?) C^0-approx: 

motivates

Def K' squeezes onto $K \subset (M^{2n+1}, \xi)$ if $\exists \epsilon(t) \downarrow 0$ s.t. $\forall t \gg 0$ $\varphi^t(K') \subset B_{\epsilon(t)}(K)$

& $\varphi^t(K')$ smoothly isotopic to K in $B_{\epsilon(t)}(K)$
for all $t \gg 0$

<u>DR '22</u> Let $N \subset M^3$ a non-leg knot
$\Lambda \subset M^3$ is leg

1. $\exists$ transverse knot T in small tub nbhd N s.t. N squeezes onto T
2. Λ does not squeeze onto N.