Beyond Semitoric

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Symplectic Zoominar
Integrable Systems

An **integrable system** is

- a $2n$-dimensional symplectic manifold $(M, \omega)$, and
- a function $F = (f_1, \ldots, f_n): M \to \mathbb{R}^n$ so that
  - $\{f_i, f_j\} = 0$ for all $i, j$, and
  - $F$ is regular on a dense set.

**Example**

- $M = \mathbb{C}$ and $F(z) = \Im(z)$. [Regular]
- $M = \mathbb{C}$ and $F(z) = |z|^2$. [Elliptic]
- $M = \mathbb{C}$ and $F(z) = \Im(z^2)$. [Hyperbolic]
- $M = \mathbb{C}^2$ and $F(x, y) = (|x|^2 - |y|^2, \Im(xy))$. [Focus-Focus]
- Any product of the first two examples. [Toric]

Here, $\omega \in \Omega^2(\mathbb{C})$ is $\sqrt{-1} dz \wedge d\bar{z}$, and $\Im(x + \sqrt{-1}y) = y$. 
Semitoric Systems

Each integrable system \((M, \omega, F)\) generates an \(\mathbb{R}^n\) action with

\[
dF_i = -\iota_{\xi_i} \omega \quad \forall \ i.
\]

\((M, \omega, F)\) is toric if \(F\) generates an \((S^1)^n\) action.

Note: In this case, every point has toric type.
Here, two points with isomorphic neighborhoods have same type.

A 4-dimensional integrable system \((M, \omega, F = (\Phi, g))\) is semitoric if

- \(\Phi\) generates an \(S^1\) action, and
- every point either
  - has toric type, or
  - has focus-focus type.

Note: This agrees with the usual definition by work of Eliasson, Vu Ngoc & Waceux, Chaperon, and Miranda & Zung.
Semitoric Systems: Results

Theorem (Vu Ngoc)

If \((M, \omega, F)\) is compact and semitoric, then \(F^{-1}(\eta, c)\) is connected for all \(\eta \in \mathbb{R}, c \in \mathbb{R}\).

Further results

- Complete Classification [Pelayo & Vu Ngoc; Palmer, Paleyo, & Tang]
- Minimal models [Kane, Palmer & Paleyo]
- Progress on Quantization [Le Floch, Paleyo & Vu Ngoc]

Pros:

- Semitoric systems are well understood, and
- there are many interesting examples.

Cons:

- We want more examples.
Examples: Toric

Example

Let $M = S^2 \times S^2$ with the product symplectic form. Let $\Phi$ be the moment map for rotating the first component. $M$ has two fixed spheres.

Claim: $M$ is (secretly) toric.
Examples: Semitoric but not toric

Example

Let $\hat{M}$ be the blowup of $M$ at three points in one fixed sphere. $\hat{M}$ has two fixed spheres and three fixed points $p_1, p_2, p_3$. If we blow up by the same amount, $\hat{\Phi}(p_1) = \hat{\Phi}(p_2) = \hat{\Phi}(p_3)$.

Claim: $\hat{M}$ is not toric [Karshon]. But it is semitoric [Hohloch, Sabatini, Sepe].
Examples: Not Semitoric

Example

Let $\tilde{M}$ be the blowup of $\hat{M}$ at $p_1$, $p_2$, and $p_3$ by the same amount. $\tilde{M}$ has two fixed spheres six fixed points, and three spheres fixed by $\mathbb{Z}_2$ with the same moment image.

Claim: $\tilde{M}$ is not semitoric [Hohloch, Sabatini, Sepe, Symington].
Complexity one spaces

A complexity one space is

- a $2n$-dimensional symplectic manifold $(M, \omega)$, and
- an effective $T := (S^1)^{n-1}$ action with moment map $\Phi: M \to \mathbb{R}^{n-1}$.

Example

Identify $H \subseteq T$ with a subgroup of $\text{SU}(h + 1)$, where $h = \dim H$.
Let $H$ act linearly on $\mathbb{C}^{h+1}$ with moment map $\Phi_H: \mathbb{C}^{h+1} \to \mathfrak{h}^\ast$.
Define a local model $Y = T \times_H \mathfrak{h}^\circ \times \mathbb{C}^{h+1}$.
There's an invariant symplectic $\omega \in \Omega^2(Y)$.
The $T$ action on $Y$ has moment map $\Phi([t, \eta, z]) = \eta + \Phi_H(z)$.

$Y$ is tall if $Y//T := \Phi^{-1}(0)/T$ contains more than one point.

Claim: Every orbit has a neighborhood isomorphic to a neighborhood of $[t, 0, 0]$ in some $Y$. 
Defining polynomials

Lemma (Karshon-T)

If $Y$ is tall, there exists $\xi \in \mathbb{Z}^{h+1}_{\geq 0}$ so that the defining polynomial

$$P([t, \eta, z]) = \prod z_i^{\xi_i}$$

induces a homeomorphism from $Y \sslash T$ to $\mathbb{C}$; $P$ has degree $N := \sum_i \xi_i$

Example

- $Y = \mathbb{C}^2$, $\lambda \cdot (x, y) = (\lambda^p x, \lambda^{-q} y)$ for $p, q > 0$, $\Phi(x, y) = px^2 - qy^2$, $P(x, y) = x^q y^p$.

- $Y = \mathbb{C}^3$, $(\alpha, \beta) \cdot (x, y, z) = (\alpha \beta x, \alpha^{-1} y, \beta^{-1} z)$, $\Phi(x, y, z) = (|x|^2 - |y|^2, |x|^2 - |z|^2)$, $P(x, y, z) = xyz$.

- $Y = S^1 \times \mathbb{Z}_2 \times \mathbb{R} \times \mathbb{C}$, $\lambda \cdot [\alpha, \eta, z] = [\lambda \alpha, \eta, z]$, $\Phi([\lambda, \eta, z]) = \eta$, $P([\lambda, \eta, z]) = z^2$. 
Ephemeral critical points

Let \( g : Y \to \mathbb{R} \) be \( T \) invariant, and let \( p = [1, 0, 0] \).

Given \( \ell \geq 0 \), let \( T^\ell_p g \) be the degree \( \ell \) Taylor polynomial of \( g \) at \( p \).

This induces a function \( T^\ell_p g : Y \parallel T \to \mathbb{R} \).

**Definition**

A point \( p \in Y \) is an **ephemeral critical point** of \( g \) if

- \( N > 1 \),
- \( T^N_p g = 0 \), and
- The zero set of \( T^N_p g \) is homeomorphic to \( \mathbb{R} \).

**Example**

- \( Y = \mathbb{C}^2 \), \( \Phi(x, y) = |x|^2 - |y|^2 \), \( g(x, y) = \Im(xy) \).
- \( Y = \mathbb{C}^2 \), \( \Phi(x, y) = p|x|^2 - q|y|^2 \), \( g(x, y) = \Im(x^q y^p) \).
- \( Y = \mathbb{C}^3 \), \( \Phi(x, y, z) = (|x|^2 - |y|^2, |x|^2 - |z|^2) \), \( g(x, y, z) = \Im(xyz) \).
- \( Y = S^1 \times \mathbb{Z}_2 \times \mathbb{R} \times \mathbb{C} \), \( \Phi([\lambda, \eta, z]) = \eta \), \( g([\lambda, \eta, z]) = \Im(z^2) \).
Near Toric

A completely integrable system \((M, \omega, F = (\Phi, g))\) is near toric if
- \(\Phi\) is the moment map of a complexity one \(T\) action, and
- every point either
  - has toric type, or
  - is an ephemeral critical point of \(g\).

Example

Every semitoric system is near toric.

Theorem (Sepe-T)

If \((M, \omega, F)\) is compact and near toric, then \(F^{-1}(\eta, c)\) is connected for all \(\eta \in \mathfrak{t}^*, c \in \mathbb{R}\).
Proof

“Proof”.

WLOG $M/(S^1)^{n-1} := \Phi^{-1}(\eta)/(S^1)^{n-1}$ contains more than one point. So it’s a closed, oriented surface $\Sigma$ with induced function $\bar{g} : \Sigma \to \mathbb{R}$. There’s a smooth structure on $\Sigma$ so that $\bar{g}$ is a Morse function. Ephemeral critical points become regular points. Points of toric type become regular points or critical points of index 0 or 2. Since $\bar{g}$ has no points of index 1, $\bar{g}^{-1}(c)$ is connected. Hence, $F^{-1}(\eta, c) = \Phi^{-1}(\eta) \cap g^{-1}(c)$ is connected.

Bonus “proofs”: $\hat{M}$ isn’t toric, $\tilde{M}$ isn’t semitoric & $M/(S^1)^{n-1} \sim S^2$.

Choose $\eta$ in the interior of $\Phi(M)$. Fixed points of focus-focus type must have weights $+1$ and $-1$. Also fixed points of toric type must become critical points of $\bar{g}$. Since $\bar{g}$ has no points of index 1, it has two critical points. Therefore, $\Phi^{-1}(\eta)$ has at most two orbits of toric type that aren’t free.
Additional Claims and Questions

Claim: Let \((M, \omega, F = (\Phi, g))\) be an integrable system such that
- \(\Phi\) is the moment map of a complexity one \(T\) action, and
- every critical point of \(F\) is non-degenerate with no hyperbolic blocks.

Then \((M, \omega, F)\) is near toric.

Note: Wacheux studied these integrable systems.

Claim: There’s a coordinate free definition of “ephemeral” using jets which is easier to check and shows that it doesn’t depend on

Claim: Our main theorem holds whenever \(\Phi\) is proper.

Question: Is every complexity one space of genus 0 a near toric system?