Beyond Semitoric

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Integrable Systems

An **integrable system** is

- a $2n$-dimensional symplectic manifold (M, ω), and
- a function $F = (f_1, \ldots, f_n)\colon M \to \mathbb{R}^n$ so that
  - $\{f_i, f_j\} = 0$ for all i, j, and
  - F is regular on a dense set.

Example

- $M = \mathbb{C}$ and $F(z) = \Im(z)$. [Regular]
- $M = \mathbb{C}$ and $F(z) = |z|^2$. [Elliptic]
- $M = \mathbb{C}$ and $F(z) = \Im(z^2)$. [Hyperbolic]
- $M = \mathbb{C}^2$ and $F(x, y) = (|x|^2 - |y|^2, \Im(xy))$. [Focus-Focus]
- Any product of the first two examples. [Toric]

Here, $\omega \in \Omega^2(\mathbb{C})$ is $\sqrt{-1}dz \wedge d\bar{z}$, and $\Im(x + \sqrt{-1}y) = y$.

Semitoric Systems

Each integrable system $(M, \omega.F)$ generates an $\mathbb{R}^n$ action with

$$dF_i = -\iota_{\xi_i}\omega \quad \forall\ i.$$

(M, ω, F) is **toric** if F generates an $(S^1)^n$ action.

Note: In this case, every point has toric type.
Here, two points with isomorphic neighborhoods have same **type**.

A 4-dimensional integrable system $(M, \omega, F = (\Phi, g))$ is **semitoric** if

- Φ generates an S^1 action, and
- every point either
  - has toric type, or
  - has focus-focus type.

Note: This agrees with the usual definition by work of Eliasson, Vu Ngoc & Waceux, Chaperon, and Miranda & Zung.

Semitoric Systems: Results

Theorem (Vu Ngoc)

If (M, ω, F) is compact and semitoric, then $F^{-1}(\eta, c)$ is connected for all $\eta \in \mathbb{R}$, $c \in \mathbb{R}$.

Further results

- Complete Classification [Pelayo & Vu Ngoc; Palmer, Paleyo, & Tang]
- Minimal models [Kane, Palmer & Peleyo]
- Progress on Quantization [Le Floch, Peleyo & Vu Ngoc]

Pros:

- Semitoric systems are well understood, and
- there are many interesting examples.

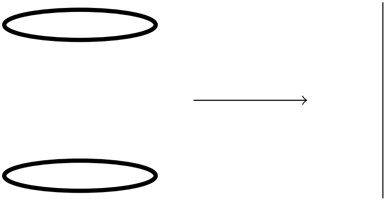
Cons:

- We want *more* examples.

Examples: Toric

Example

Let $M = S^2 \times S^2$ with the product symplectic form.
Let Φ be the moment map for rotating the first component.
M has two fixed spheres.

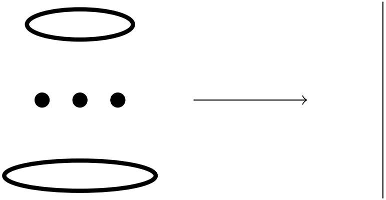


Claim: M is (secretly) toric.

Examples: Semitoric but not toric

Example

Let $\widehat{M}$ be the blowup of M at three points in one fixed sphere.
$\widehat{M}$ has two fixed spheres and three fixed points p_1, p_2, p_3.
If we blow up by the same amount, $\widehat{\Phi}(p_1) = \widehat{\Phi}(p_2) = \widehat{\Phi}(p_3)$.

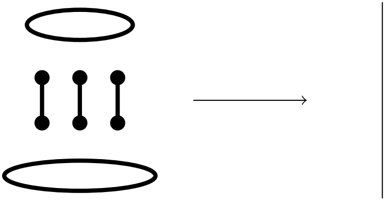


Claim: $\widehat{M}$ is not toric [Karshon].
But it is semitoric [Hohloch, Sabatini, Sepe].

Examples: Not Semitoric

Example

Let $\widetilde{M}$ be the blowup of $\widehat{M}$ at p_1, p_2, and p_3 by the same amount.
$\widetilde{M}$ has two fixed spheres six fixed points, and three spheres fixed by $\mathbb{Z}_2$ with the same moment image.



Claim: $\widetilde{M}$ is not semitoric [Hohloch, Sabatini, Sepe, Symington].

Complexity one spaces

A **complexity one space** is

- a $2n$-dimensional symplectic manifold (M, ω), and
- an effective $T := (S^1)^{n-1}$ action with moment map $\Phi \colon M \to \mathbb{R}^{n-1}$.

Example

Identify $H \subseteq T$ with a subgroup of $\mathrm{SU}(h+1)$, where $h = \dim H$.
Let H act linearly on $\mathbb{C}^{h+1}$ with moment map $\Phi_H \colon \mathbb{C}^{h+1} \to \mathfrak{h}^*$.
Define a **local model** $Y = T \times_H \mathfrak{h}^\circ \times \mathbb{C}^{h+1}$.
There's an invariant symplectic $\omega \in \Omega^2(Y)$.
The T action on Y has moment map $\Phi([t, \eta, z]) = \eta + \Phi_H(z)$.

Y is **tall** if $Y /\!/ T := \Phi^{-1}(0)/T$ contains more than one point.

Claim: Every orbit has a neighborhood isomorphic to a neighborhood of $[t, 0, 0]$ in some Y.

Defining polynomials

Lemma (Karshon-T)

If Y is tall, there exists $\xi \in \mathbb{Z}_{\geq 0}^{h+1}$ so that the **defining polynomial**

$$P([t, \eta, z]) = \prod z_i^{\xi_i}$$

induces a homeomorphism from $Y /\!/ T$ to $\mathbb{C}$; P has **degree** $N := \sum_i \xi_i$

Example

- $Y = \mathbb{C}^2$, $\lambda \cdot (x, y) = (\lambda^p x, \lambda^{-q} y)$ for $p, q > 0$,
  $\Phi(x, y) = p|x|^2 - q|y|^2$, $P(x, y) = x^q y^p$.
- $Y = \mathbb{C}^3$, $(\alpha, \beta) \cdot (x, y, z) = (\alpha\beta x, \alpha^{-1} y, \beta^{-1} z)$,
  $\Phi(x, y, z) = (|x|^2 - |y|^2, |x|^2 - |z|^2)$, $P(x, y, z) = xyz$.
- $Y = S^1 \times_{\mathbb{Z}_2} \mathbb{R} \times \mathbb{C}$, $\lambda \cdot [\alpha, \eta, z] = [\lambda\alpha, \eta, z]$,
  $\Phi([\lambda, \eta, z]) = \eta$, $P([\lambda, \eta, z]) = z^2$.

Ephemeral critical points

Let $g\colon Y \to \mathbb{R}$ be T invariant, and let $p = [1, 0, 0]$.
Given $\ell \geq 0$, let $T_p^\ell g$ be the degree ℓ **Taylor polynomial** of g at p.
This induces a function $T_p^\ell \overline{g}\colon Y/\!/T \to R$.

Definition

A point $p \in Y$ is an **ephemeral critical point** of g if

- $N > 1$,
- $T_p^{N-1}\overline{g} = 0$, and
- The zero set of $T_p^N \overline{g}$ is homeomorphic to $\mathbb{R}$.

Example

- $Y = \mathbb{C}^2$, $\Phi(x, y) = |x|^2 - |y|^2$, $g(x, y) = \Im(xy)$.
- $Y = \mathbb{C}^2$, $\Phi(x, y) = p|x|^2 - q|y|^2$, $g(x, y) = \Im(x^q y^p)$.
- $Y = \mathbb{C}^3$, $\Phi(x, y, z) = (|x|^2 - |y|^2, |x|^2 - |z|^2)$, $g(x, y, z) = \Im(xyz)$
- $Y = S^1 \times_{\mathbb{Z}_2} \mathbb{R} \times \mathbb{C}$, $\Phi([\lambda, \eta, z]) = \eta$, $g([\lambda, \eta, z]) = \Im(z^2)$.

Near Toric

A completely integrable system $(M, \omega, F = (\Phi, g))$ is **near toric** if

- Φ is the moment map of a complexity one T action, and
- every point either
  - has toric type, or
  - is an ephemeral critical point of g.

Example

Every semitoric system is near toric.

Theorem (Sepe-T)

If (M, ω, F) is compact and near toric, then $F^{-1}(\eta, c)$ is connected for all $\eta \in \mathfrak{t}^*, c \in \mathbb{R}$.

Proof

"Proof".

WLOG $M//(S^1)^{n-1} := \Phi^{-1}(\eta)/(S^1)^{n-1}$ contains more than one point.
So it's a closed, oriented surface Σ with induced function $\overline{g}\colon \Sigma \to \mathbb{R}$.
There's a smooth structure on Σ so that $\overline{g}$ is a Morse function.
Ephemeral critical points become regular points.
Points of toric type become regular points or critical points of index 0 or 2.
Since $\overline{g}$ has no points of index 1, $\overline{g}^{-1}(c)$ is connected.
Hence, $F^{-1}(\eta, c) = \Phi^{-1}(\eta) \cap g^{-1}(c)$ is connected.

□

Bonus "proofs": $\widehat{M}$ isn't toric, $\widetilde{M}$ isn't semitoric & $M//(S^1)^{n-1} \simeq S^2$.

Choose η in the interior of $\Phi(M)$.
Fixed points of focus-focus type must have weights $+1$ and -1.
Also fixed points of toric type must become critical points of $\overline{g}$.
Since $\overline{g}$ has no points of index 1, it has two critical points.
Therefore, $\Phi^{-1}(\eta)$ has at most two orbits of toric type that aren't free. □

Additional Claims and Questions

Claim: Let $(M, \omega, F = (\Phi, g))$ be an integrable system such that

- Φ is the moment map of a complexity one T action, and
- every critical point of F is non-degenerate with no hyperbolic blocks.

Then (M, ω, F) is near toric.

Note: Wacheux studied these integrable systems.

Claim: There's a coordinate free definition of "ephemeral" using jets which is easier to check and shows that it doesn't depend on

Claim: Our main theorem holds whenever Φ is proper.

Question: Is every complexity one space of genus 0 a near toric system?