Geometry and topology of Hamiltonian Floer complexes in low-dimension
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Motivating question

Let (H, J) be such that we may define $CF_*(H, J)$. What dynamical information about the isotopy (ϕ_t^H) generated by H can be extracted from the (filtered) Floer complexes $CF_*(H, J)$?

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We just don't really understand solutions to Floer's equation all that well, from a geometric point of view.

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- positivity of intersections for pseudo-holomorphic curves
- Ideas originating in the low-dimensional contact setting by Hofer-Wysocki-Zehnder (‘95,‘96,‘99,‘03) and Siefring (‘08)

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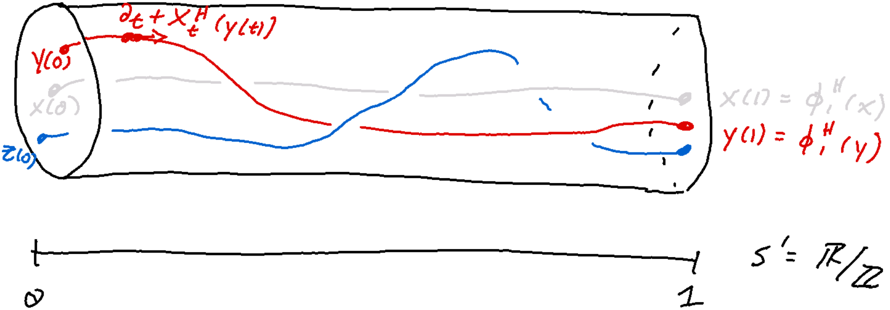

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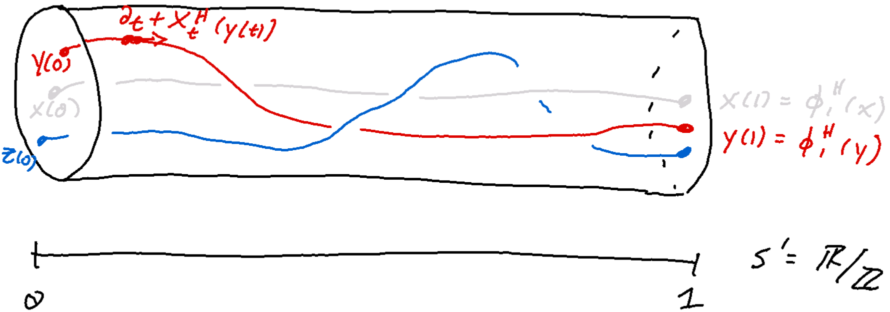

Think of the graphs of the 1-periodic orbits in $Per_0(H)$ as forming a braid P^H in $S^1 \times \Sigma$, and we'd like to ask how the topology of the indexed braid (P^H, μ_{CZ}) relates to the structure of $CF_*(H, J)$ for generic J

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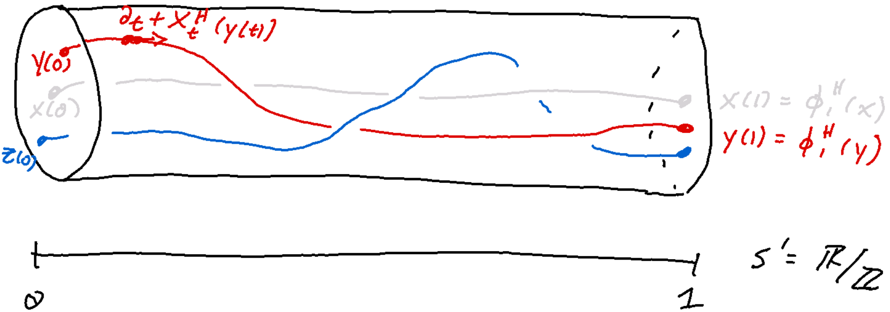

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Braids and capped braids

For our purposes a **braid** $X = \{x_i\}_{i=1}^k$ is a finite collection of smooth loops $x_i \in \mathcal{L}(\Sigma)$ such that the images of the graphs $t \mapsto \check{x}_i(t) := (t, x_i(t))$, $t \in S^1$ are all pairwise disjoint.

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Remark: Braid theorists would tend to call these something like *(closed) geometric pure braids*.

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- **unlinked** if there exist choices of cappings $w_i : D^2 \to \Sigma$ with $[w_i] = \alpha_i \in \pi_2(\Sigma; x_i)$ for each $i = 1, \ldots, k$ such that the images of the graphs $\tilde{w}_i(z) = (z, w_i(z))$ are all pairwise disjoint in $D^2 \times \Sigma$.

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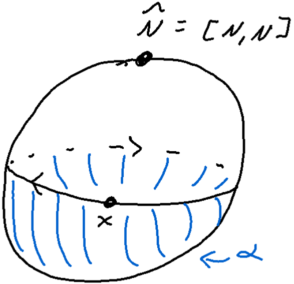

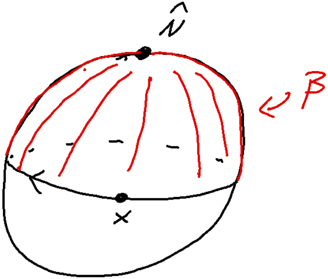

$\hat{X} = \{\hat{N}, [x, \alpha]\}$
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$\hat{Y} = \{\hat{N}, [x, \beta]\}$
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- **positive** (resp. **negative**) if there exist choices of cappings $w_i : D^2 \to \Sigma$ with $[w_i] = \alpha_i \in \pi_2(\Sigma; x_i)$ for each $i = 1, \dots, k$ such that the images of the graphs $\tilde{w}_i(z) = (z, w_i(z))$ are all pairwise transverse in $D^2 \times \Sigma$ and all intersections are positive (resp. negative).

Chain-level PSS maps

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by considering the map on homology induced by the **chain-level PSS maps** of the form

$$\Phi_{\mathcal{D}}^{PSS} : QC_*(f,g) = (C^{Morse}(f,g) \otimes \Lambda_\omega)_* \to CF_{*-n}(H,J)$$

depending on generic **regular PSS data** $\mathcal{D} = (f, g; \mathcal{H}, \mathbb{J}) \in PSS^{reg}(H, J)$

Question: Given (H, J) and $\alpha \in QH_*(M)$, $\alpha \neq 0$, can you find a Floer cycle which represents α? (In Morse theory, easy. In Floer theory: not so easy)

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Theorem A

Let $\sigma \in CF_1(H, J)$ (with $\mathbb{Z}/2Z$-coefficients). Then σ is a non-trivial cycle such that $\sigma \in im\, \Phi_{\mathcal{D}}^{PSS}$ for some chain-level PSS map $\Phi_{\mathcal{D}}^{PSS}$ if and only if supp σ is a maximal positive capped braid relative index 1.

Definition

A capped braid $\hat{X} \subset \widetilde{Per_0(H)}$ is **maximally positive relative index** 1 if $\mu_{CZ}(\hat{x}) = 1$ for all $\hat{x} \in \hat{X}$, $\hat{X}$ is a positive capped braid, and $\hat{X}$ is maximal with respect to these two conditions.

Write $mp_{(1)}(H)$ for the set of all such capped braids.

A novel family of spectral invariants

Definition

Let (H, J) be Floer-regular. For $\alpha \in QH_*(M, \omega)$, $\alpha \neq 0$, we define the **PSS-image spectral invariant**

$$c_{im}(\alpha; H, J) :=$$
$$\inf_{\mathcal{D} \in PSS_{reg}(H,J)} \{\lambda_H(\sigma) : \sigma \in \text{im } \Phi_{\mathcal{D}}^{PSS}, \text{ such that } [\sigma] = (\Phi_{\mathcal{D}}^{PSS})_* \alpha\}$$

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Independent of J, so write $c_{im}(\alpha; H)$.

Basic properties of the PSS-image spectral invariants

For any $H, F \in C^\infty(S^1 \times M)$ and any $\alpha \in QH_*(M, \omega) \setminus \{0\}$

1. $c_{OS}(\alpha; H) \leq c_{im}(\alpha; H)$
2. (Non-degenerate spectrality) $c_{im}(\alpha; H) \in Spec(H)$, for all non-degenerate H
3. (Hofer continuity) $|c_{im}(\alpha; H) - c_{im}(\alpha; F)| \leq \|H - F\|_{L^{1,\infty}}$
4. (Symplectic invariance) $c_{im}(\alpha; \psi^* H) = c_{im}(\alpha; H)$ for any symplectomorphism ψ
5. (Projective invariance) For any $\lambda \in \mathbb{Q}$, $\lambda \neq 0$, $c_{im}(\lambda\alpha; H) = c_{im}(\alpha; H)$
6. (Normalization) For $\alpha = \sum \alpha_A e^A \in QH_*(M) \setminus \{0\}$
$$c_{im}(\alpha; 0) = \max\{-\omega(A) : \alpha_A \neq 0\}$$
7. (**Weak** triangle inequality)
$$c_{im}(\alpha; H\#F) \leq c_{im}(\alpha; H) + c_{im}([M]; F)$$

Basic properties of the PSS-image spectral invariants

Moral: Most arguments using Oh-Schwarz spectral invariants adapt straight-forwardly to give arguments for PSS-image spectral invariants. (Especially when $\alpha = [M]$).

Some interesting consequences of Theorem A

Corollary

Let H be non-degenerate.

$$c_{im}([\Sigma]; H) = \inf_{\hat{X} \in mp_{(1)}(H)} \sup_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x}).$$

Corollary

On surfaces, the symplectically bi-invariant norm γ_{im} is both C^0-continuous and Hofer-continuous. Moreover, if H is non-degenerate, then

$$\gamma_{im}(\phi) = \inf_{\hat{X} \in mp_{(1)}(H)} \sup_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x}) - \sup_{\hat{X} \in mn_{(-1)}(H)} \inf_{\hat{x} \in \hat{X}} \mathcal{A}_H(\hat{x}),$$

for H any normalized Hamiltonian such that $\phi_1^H = \phi$.

Question: What can Floer theory tell us about "how to think about Hamiltonians geometrically/topologically" in a similar way to how Morse theory tells us "how to think about functions geometrically/topologically"?

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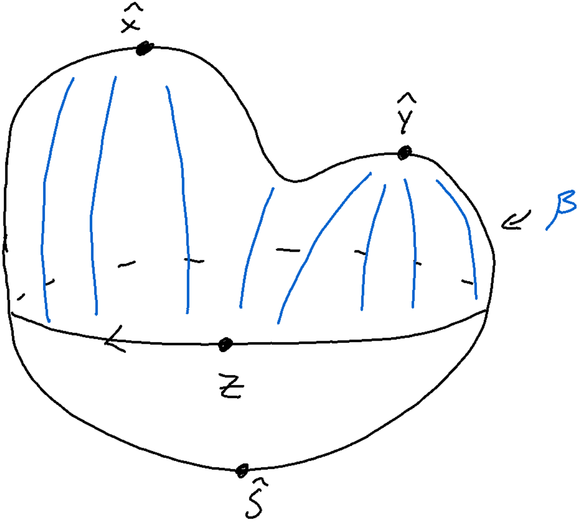
Definition

A capped braid $\hat{X} \subset \widetilde{Per_0(H)}$ is **maximally unlinked relative the Morse range** (or **murm**) if $\mu_{CZ}(\hat{x}) \in \{-1, 0, 1\}$ for all $\hat{x} \in \hat{X}$, $\hat{X}$ is an unlinked capped braid, and $\hat{X}$ is maximal with respect to these two conditions.

Write $\hat{X} \in murm(H)$ for the set of all such capped braids.



If

$\mu_{CZ}([z, \beta]) = 1$

$\mu_{CZ}(\hat{S}) = -1$

then

$\{[z, \beta], \hat{S}\}$

is nurm.

Theorem B

Let $H \in C^\infty(S^1 \times \Sigma)$ be a non-degenerate Hamiltonian, and let $J \in C^\infty(S^1; \mathcal{J}_\omega(\Sigma))$ be such that (H, J) is Floer regular. For any capped braid $\hat{X} \in murm(H)$, we may construct an oriented singular foliation $\mathcal{F}^{\hat{X}}$ of $S^1 \times \Sigma$ with the following properties

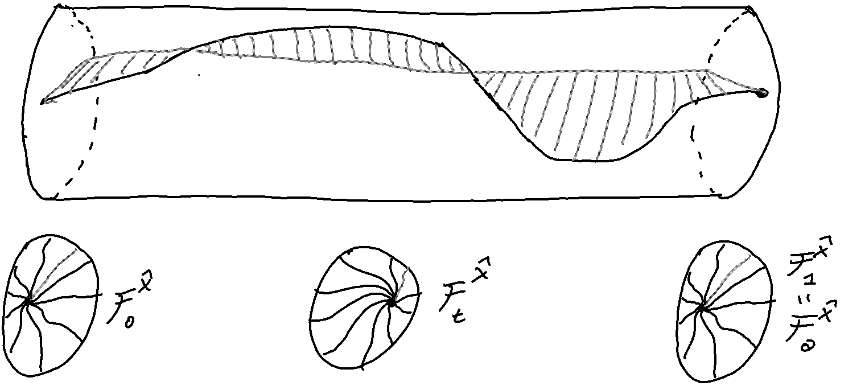
1. *The singular leaves of $\mathcal{F}^{\hat{X}}$ are precisely the graphs of the orbits in $\hat{X}$.*
2. *The regular leaves are annuli parametrized by maps*

$$\begin{aligned} \check{u} : \mathbb{R} \times S^1 &\to S^1 \times \Sigma \\ (s, t) &\mapsto (t, u(s, t)). \end{aligned}$$

   for $u \in \widetilde{\mathcal{M}}(\hat{x}, \hat{y}; H, J)$, for $\hat{x}, \hat{y} \in \hat{X}$.

3. *The vector field $\check{X}^H(t, z) = \partial_t \oplus X_t^H(z)$ is positively transverse to every regular leaf of $\mathcal{F}^{\hat{X}}$.*

A single leaf of the foliation $\mathcal{F}^{\hat{x}}$, with $\hat{x} = ([z, \beta], \hat{s})$



Some consequences of Theorem B

For any $\hat{X} \in murm(H)$ we can define the $\hat{X}$*-restricted action functional*

$$\begin{aligned} A^{\hat{X}} &: S^1 \times \Sigma \to \mathbb{R} \\ &(t, u(s,t)) \mapsto \mathcal{A}_H(\hat{u}_s). \end{aligned}$$

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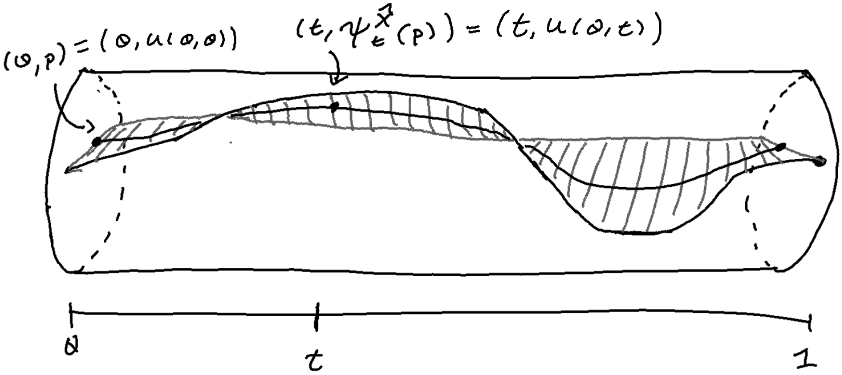
- Define $A_t^{\hat{X}} \in C^\infty(\Sigma)$, $t \in S^1$, to be its restriction to the fiber $\{t\} \times \Sigma$.
- Obtain an S^1-family of Morse functions, such that the negative gradient flow of $A_t^{\hat{X}}$ provides a singular foliation which coincides with the foliation $\mathcal{F}_t^{\hat{X}}$ given by intersecting $\mathcal{F}^{\hat{X}}$ with the fiber over $t \in S^1$.

Some consequences of Theorem B

We also obtain a loop of diffeomorphisms

$$\psi_t^{\hat{X}}(u(0,0)) := u(t,0), t \in S^1$$

given by "sliding in the $\partial_t u$-direction".



Corollary

For every $t \in S^1$, $\mathcal{F}_t^{\hat{X}}$ is a singular foliation of Morse type. Moreover, the loop $(\psi_t^{\hat{X}})_{t\in S^1}$ is a contractible loop such that the orbits of $(\psi^{\hat{X}})^{-1} \circ \phi^H$ are positively transverse to the foliation $\mathcal{F}_0^{\hat{X}}$.

Some consequences of Theorem B

Turns out that any two capped orbits $\hat{x}, \hat{y} \in \widetilde{Per_0}(H)$ with $\mu_{CZ}(\hat{x}), \mu_{CZ}(\hat{y}) \in \{-1, 0, 1\}$ must be unlinked if they are connected by a Floer cylinder

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Shows that the problem of understanding $CF_*(H, J)$ in the range where $* \in \{-1, 0, 1\}$ is equivalent to the problem of understanding the finitely many Morse functions $A_0^{\hat{X}}$ for $\hat{X} \in murm(H)$ together with the combinatorics of the set $murm(H)$.

Some consequences of Theorem B

Let H be non-degenerate, $\hat{X} = \{[x_i(t), w_i(se^{2\pi it})]\}_{i=1}^{k} \in murm(H)$ and for any $m \in \mathbb{Z}_{>0}$, denote by $\hat{X}^{\sharp m} = \{[x_i(mt), w_i(se^{2m\pi it})]\}_{i=1}^{k} \subset \widetilde{Per}_0(H^{\sharp m})$ its m-fold iterate.

Corollary

For any $\hat{y} \in \widetilde{Per}_0(H^{\sharp m})$ with $y(t) \neq x_i(mt)$ for all $i = 1, \dots, k$ and $t \in S^1$, $\hat{X}^{\sharp m} \cup \hat{y}$ is linked. In particular every $\hat{X} \in murm(H)$ is maximally unlinked as a subset of $\widetilde{Per}_0(H)$.

Some consequences of Theorem B

Le Calvez ('05) associate 'homotopically positively transverse' oriented singular (continuous) foliations on Σ to each 'maximal isotopy' from *id* to $\phi \in Homeo_0(\Sigma)$ by developing 'equivariant Brouwer theory'.

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In fact, the Floer-theoretic construction provides instances of foliations which are 'even nicer than expected' from the perspective of Le Calvez's theory.

Thank You!