Higher algebra of $A_\infty$-algebras in Morse theory

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The results presented in this talk are taken from my two recent papers: *Higher algebra of $A_\infty$ and $\Omega$BA$s$-algebras in Morse theory I* (arXiv:2102.06654) and *Higher algebra of $A_\infty$ and $\Omega$BA$s$-algebras in Morse theory II* (arxiv:2102.08996).
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Definition

Let $A$ be a cochain complex with differential $m_1$. An $A_\infty$-algebra structure on $A$ is the data of a collection of maps of degree $2 - n$

$$m_n : A^\otimes n \to A, \ n \geq 1,$$

extending $m_1$ and which satisfy

$$[m_1, m_n] = \sum_{i_1 + i_2 + i_3 = n} \pm m_{i_1+1+i_3}(\text{id}^\otimes i_1 \otimes m_{i_2} \otimes \text{id}^\otimes i_3).$$

These equations are called the $A_\infty$-equations.
Representing $m_n$ as $\begin{array}{c} ^{1,2} \infty ^n \end{array}$, these equations can be written as

$$[m_1, \begin{array}{c} ^{1,2} \infty ^n \end{array}] = \sum_{\begin{array}{c} h+k=n+1 \\ 2 \leq h \leq n-1 \\ 1 \leq i \leq k \end{array}} \pm \begin{array}{c} ^1 \infty \ldots ^1 \infty ^k \\ \bullet \ldots \bullet \ldots \bullet \end{array}.$$
In particular,

\[ [m_1, m_2] = 0, \]
\[ [m_1, m_3] = m_2(id \otimes m_2 - m_2 \otimes id), \]

implying that \( m_2 \) descends to an associative product on \( H^*(A) \). An \( A_\infty \)-algebra is thus simply a correct notion of a dg-algebra whose product is associative up to homotopy.

The operations \( m_n \) are the higher coherent homotopies which keep track of the fact that the product is associative up to homotopy.
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Definition

An $A_\infty$-morphism between two $A_\infty$-algebras $A$ and $B$ is a family of maps $f_n : A^\otimes n \to B$ of degree $1 - n$ satisfying

$$[m_1, f_n] = \sum_{i_1 + i_2 + i_3 = n, \ i_2 \geq 2} \pm f_{i_1 + 1 + i_2} (\text{id}^\otimes i_1 \otimes m_{i_2} \otimes \text{id}^\otimes i_3)$$

$$+ \sum_{i_1 + \cdots + i_s = n, \ s \geq 2} \pm m_s (f_{i_1} \otimes \cdots \otimes f_{i_s}).$$
Representing the operations $f_n$ as $\text{blue}$, the operations $m_n^B$ in red and the operations $m_n^A$ in blue, these equations read as

$$[m_1, \text{blue}] = \sum_{h+k=n+1\atop 1<i<k\atop h\geq 2} \pm 1 \text{blue} + \sum_{i_1+\cdots+i_s=n\atop s\geq 2} \pm \text{red}.$$
We check that

\[ [m_1, f_1] = 0 \]

\[ [m_1, f_2] = f_1 m_2^A - m_2^B (f_1 \otimes f_1) . \]

An \( A_\infty \)-morphism between \( A_\infty \)-algebras induces a morphism of associative algebras on the level of cohomology, and is a correct notion of morphism which preserves the product up to homotopy.
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Our goal now: study the *higher algebra of $A_\infty$-algebras*.

Considering two $A_\infty$-morphisms $F$, $G$, we would like first to determine a notion giving a satisfactory meaning to the sentence "$F$ and $G$ are homotopic". Then, $A_\infty$-homotopies being defined, what is now a good notion of a homotopy between homotopies? And of a homotopy between two homotopies between homotopies? And so on.
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Definition ([LH02])

An $A_\infty$-homotopy between two $A_\infty$-morphisms $(f_n)_{n \geq 1}$ and $(g_n)_{n \geq 1}$ is a collection of maps

$$h_n : A^{\otimes n} \to B,$$

of degree $-n$, satisfying

$$[\partial, h_n] = g_n - f_n + \sum_{i_1 + i_2 + i_3 = m \atop i_2 \geq 2} \pm h_{i_1 + 1 + i_3} (\text{id} \otimes^{i_1} \otimes m_{i_2} \otimes \text{id} \otimes^{i_3})$$

$$+ \sum_{i_1 + \cdots + i_s + l \atop + j_1 + \cdots + j_t = n \atop s + 1 + t \geq 2} \pm m_{s+1+t} (f_{i_1} \otimes \cdots \otimes f_{i_s} \otimes h_l \otimes g_{j_1} \otimes \cdots \otimes g_{j_t}).$$
In symbolic formalism,

\[
[\partial, \begin{array}{c} \circ \\ [0 < 1] \end{array}] = \begin{array}{c} \circ \\ [1] \end{array} - \begin{array}{c} \circ \\ [0] \end{array} + \sum \pm \begin{array}{c} \circ \\ [0 < 1] \end{array} + \sum \pm \begin{array}{c} \circ \\ [0] \end{array} \begin{array}{c} \circ \\ [0 < 1] \end{array} \begin{array}{c} \circ \\ [1] \end{array},
\]

where we denote \( \begin{array}{c} \circ \\ [0] \end{array} \), \( \begin{array}{c} \circ \\ [0 < 1] \end{array} \) and \( \begin{array}{c} \circ \\ [1] \end{array} \) respectively for the \( f_n \), the \( h_n \) and the \( g_n \).
The relation \textit{being $A_{\infty}$-homotopic} on the class of $A_{\infty}$-morphisms is an equivalence relation. It is moreover stable under composition.
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We denote the standard $n$-simplex $\Delta^n$ as $[0 < \cdots < n]$ and a subface of $\Delta^n$ as $[i_1 < \cdots < i_k]$. 
Definition ([MS03])

Let $I$ be a face of $\Delta^n$. An overlapping partition of $I$ is a sequence of faces $(I_\ell)_{1 \leq \ell \leq s}$ of $I$ such that

(i) the union of this sequence of faces is $I$, i.e. $\bigcup_{1 \leq \ell \leq s} I_\ell = I$;

(ii) for all $1 \leq \ell < s$, $\max(I_\ell) = \min(I_{\ell+1})$.

An overlapping 6-partition for $[0 < 1 < 2]$ is for instance

$$[0 < 1 < 2] = [0] \cup [0] \cup [0 < 1] \cup [1] \cup [1 < 2] \cup [2].$$
Definition ([Maz21b])

A \textit{n-morphism} from $A$ to $B$ is defined to be a collection of maps $f_{l}^{(m)} : A \otimes^{m} \rightarrow B$ of degree $1 - m - \dim(l)$ for $l \subset \Delta^{n}$ and $m \geq 1$, that satisfy

\[
\left[ \partial, f_{l}^{(m)} \right] = \sum_{j=0}^{\dim(l)} (-1)^{j} f_{\partial_{j}l}^{(m)} + \sum_{i_{1} + \cdots + i_{s} = m} \pm m_{s}(f_{i_{1}}^{(1)} \otimes \cdots \otimes f_{i_{s}}^{(s)}) \\
+ (-1)^{|l|} \sum_{i_{1} + i_{2} + i_{3} = m} \pm f_{l}^{(i_{1} + 1 + i_{3})}(\text{id} \otimes i_{1} \otimes m_{i_{2}} \otimes \text{id} \otimes i_{3}) .
\]
Equivalently and more visually, a collection of maps \( \begin{array}{c} \cdot \rightarrow \cdot \end{array} \) satisfying

\[
[\partial, \begin{array}{c} \cdot \rightarrow \cdot \end{array}] = \sum_{j=1}^{k} (-1)^j \sum_{\text{sing} j} \cdot \rightarrow \cdot + \sum_{l_1 \cup \cdots \cup l_s = l} \pm \cdot \rightarrow \cdot + \sum_{l} \pm \cdot \rightarrow \cdot
\]
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The sets $\text{HOM}_{\mathcal{A}_\infty} - \text{alg}(A, B)_n$ of $n$-morphisms from $A$ to $B$ then fit into a HOM-simplicial set $\text{HOM}_{\mathcal{A}_\infty} - \text{alg}(A, B)$. 

**Theorem ([Maz21b])**

*For $A$ and $B$ two $\mathcal{A}_\infty$-algebras, the simplicial set $\text{HOM}_{\mathcal{A}_\infty}(A, B)$ is a Kan complex.*

The simplicial homotopy groups of the Kan complex $\text{HOM}_{\mathcal{A}_\infty}(A, B)$ can moreover be explicitly computed.
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There exists a collection of polytopes, called the *associahedra* and denoted \( \{ K_n \} \), which encode the \( A_\infty \)-equations between \( A_\infty \)-algebras. This means that \( K_n \) has dimension \( n - 2 \) and that its boundary is modeled on the \( A_\infty \)-equations read as

\[
[m_1, \quad \begin{array}{c}\vdots \\
1 & 2 & \cdots & n
\end{array} ] = \sum_{\begin{array}{c}h+k=n+1 \\
2 \leq h \leq n-1 \\
1 \leq i \leq k
\end{array}} \pm \quad \begin{array}{c}1 \\
\vdots \\
d_1 \quad d_2 \quad k
\end{array}.
\]
Figure: The associahedra $K_2$, $K_3$ and $K_4$, with cells labeled by the operations they define
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There exists a collection of polytopes, called the *multiplihedra* and denoted \( \{ J_n \} \), which encode the \( A_\infty \)-equations for \( A_\infty \)-morphisms. Again, \( J_n \) has dimension \( n - 1 \) and the boundary of \( J_n \) is modeled on the \( A_\infty \)-equations for \( A_\infty \)-morphisms are

\[
\partial(\bullet) = \sum_{\substack{h+k=n+1 \\\text{and} \\ 1 \leq i \leq k \\\text{and} \\ h \geq 2}} \pm 1^{i} h^{k} + \sum_{\substack{i_1 + \cdots + i_s = n \\ s \geq 2}} \pm 1^{i_1} i^{1} \cdots i^{s}.
\]
**Figure**: The multiplihedra $J_1$, $J_2$ and $J_3$ with cells labeled by the operations they define in $A_\infty$ -- Morph
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We would like to define a family of polytopes encoding $n$-morphisms between $A_\infty$-algebras. The natural candidate is $\Delta^n \times J_m$.

We prove in [Maz21b] that there exists a refined polytopal subdivision of $\Delta^n \times J_m$ encoding the $A_\infty$-equations for $n$-morphisms between $A_\infty$-algebras. We define the $n$-multiplihedra to be the polytopes $\Delta^n \times J_m$ endowed with this polytopal subdivision and denote them $n - J_m$. 
Figure: The 1-multiplihedron $\Delta^1 \times J_2$
Figure: The 2-multiplihedron $\Delta^2 \times J_2$
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Figure: The 1-multiplihedron $\Delta^1 \times J_3$
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Let $M$ be an oriented closed Riemannian manifold endowed with a Morse function $f$ together with a Morse-Smale metric. The Morse cochains $C^*(f)$ form a deformation retract of the singular cochains $C_{sing}^*(M)$ as shown in [Hut08].

\[
    \begin{array}{cccc}
    h & : & (C_{sing}^*, \partial_{sing}) & \xrightarrow{p} & (C^*(f), \partial_{Morse}) \ \\
    & & \downarrow i & & \\
    & & (C_{sing}^*, \partial_{sing}) & \xrightarrow{p} & (C^*(f), \partial_{Morse}) \ \\
    \end{array}
\]

The cup product naturally endows the singular cochains $C_{sing}^*(M)$ with a dg-algebra structure. The homotopy transfer theorem ensures that it can be transferred to an $A_\infty$-algebra structure on the Morse cochains $C^*(f)$. 
The differential on the Morse cochains is defined by a count of moduli spaces of gradient trajectories. Is it then possible to define higher multiplications $m_n$ on $C^*(f)$ by a count of moduli spaces such that they fit in a structure of $A_\infty$-algebra?

Question solved by Abouzaid in [Abo11], drawing from earlier works by Fukaya ([Fuk97] for instance), using moduli spaces of perturbed Morse gradient trees.
We prove in [Maz21a] and [Maz21b] that given two Morse functions $f$ and $g$, one can in fact construct $n$-morphisms between their Morse cochain complexes $C^*(f)$ and $C^*(g)$ through a count of geometric moduli spaces of perturbed Morse gradient trees.

These constructions stem from the fact that ...
... the associahedra can be realized as the compactified moduli spaces of stable metric ribbon trees ...

**Figure:** The compactified moduli space $\overline{T}_4$
... and the multiplihedra can be realized as the compactified moduli spaces of stable two-colored metric ribbon trees.

The compactified moduli space $\overline{CT}_3$
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1. It is quite clear that given two compact symplectic manifolds $M$ and $N$, one should be able to construct $n$-morphisms between their Fukaya categories $\text{Fuk}(M)$ and $\text{Fuk}(N)$ through counts of moduli spaces of quilted disks (see [MWW18] for the $n = 0$ case).
2. Given three Morse functions $f_0, f_1, f_2$ and geometrical $A_\infty$-morphisms $\mu_{ij} : C^*(f_i) \to C^*(f_j)$, can we construct an $A_\infty$-homotopy such that $\mu_{12} \circ \mu_{01} \simeq \mu_{02}$ through this homotopy?
That is, can the following diagram be filled in the $A_\infty$ realm

$$
\begin{array}{ccc}
C^*(f_0) & \xrightarrow{\mu_{01}} & C^*(f_1) \\
\downarrow{\mu_{02}} & & \downarrow{\mu_{12}} \\
C^*(f_2) & & ?
\end{array}
$$

Work in progress, see also [MWW18] for a similar question.
3. Links between the $n$-multiplihedra and the 2-associahedra of Bottman (see [Bot19a] and [Bot19b] for instance)?
Thanks for your attention!


References II