On Symplectic Capacities and their Blind Spots

with Yuanpu Liang

Symplectic capacity

$$C : \mathcal{S} \subset \{ U \subset \mathbb{R}^{2n} \} \longrightarrow [0, \infty]$$

1) $C(U) \leq C(V)$ if $\exists\ \varphi \in \mathrm{Symp}(\mathbb{R}^{2n}, \omega_{2n})$ s.t. $\varphi(U) \subset V$.

2) $C(\lambda U) = \lambda^2 C(U)$

3) $C(B^{2n}(1)) > 0$, $C(B^2(1) \times \mathbb{R}^{2n-2}) < \infty$

Examples

- Gromov width $= \sup \left\{ \pi r^2 \;\middle|\; \begin{array}{l} \exists\, \varphi \in Symp(\mathbb{R}^{2n}, \omega_{2n}) \text{ s.t.} \\ \varphi(B^{2n}(r)) \subset U \end{array} \right\}$

- $\{c_k^{EH}\}_{k \in \mathbb{N}}$ Ekeland - Hofer capacities

Defined for all subsets of $\mathbb{R}^{2n}$ in terms of periodic orbits of autonomous Hamiltonians.

Gutt-Hutchings capacities

$$c_k, \quad k \in \mathbb{N}$$

Defined for star-shaped subsets $X \subset \mathbb{R}^{2n}$

using S^1-equivariant symplectic homology

$$CH_*(X) = \begin{cases} \mathbb{Q}, & * \in n-1+2\mathbb{N} \\ 0, & \text{otherwise} \end{cases}$$

$$c_k(X) = \inf \left\{ L \;\middle|\; \begin{array}{l} \text{image } \; i_L : CH^L(X) \to CH(X) \\ \text{contains } \; CH_{n-1+2k}(X). \end{array} \right\}$$

Conjecture (Gutt-Hutchings)

$$c_k(X) = c_k^{EH}(X) \quad \text{for all } X \text{ star-shaped}$$

This holds for symplectic ellipsoids and polydisks.

$$E(1,a) = \left\{ \pi|z_1|^2 + \frac{\pi|z_2|^2}{a} \leq 1 \right\} \subset \mathbb{C}^2 = \mathbb{R}^4$$

$$c_k(E(1,a)) = \left(\text{Sort}\left[\mathbb{N} \cup a\mathbb{N} \right] \right)[k] = c_k^{EH}(E(1,a))$$

$$P(1,a) = \left\{ \pi|z_1|^2 < 1 \ , \ \pi|z_2|^2 < a \right\}$$

$$c_k(P(1,a)) = k. = c_k^{EH}(P(1,a))$$

Gutt-Hutchings formulae formulae for convex/concave toric domains

$$\mu : \mathbb{C}^n = \mathbb{R}^{2n} \longrightarrow \mathbb{R}^n_{\geq 0}$$

$$(z_1, \dots, z_n) \longmapsto (\pi|z_1|^2, \dots, \pi|z_n|^2)$$

$$\Omega \subset \mathbb{R}^n_{\geq 0} \rightsquigarrow X_\Omega = \mu^{-1}(\Omega) \subset \mathbb{R}^{2n}$$

X_Ω is convex if $\hat{\Omega} = \{x \mid (|x_1|, \dots, |x_n|) \in \Omega\} \subset \mathbb{R}^n$ is convex.

X_Ω is concave if Ω^c is convex in $\mathbb{R}^n_{\geq 0}$

Theorem (Gutt-Hutchings)

I) If X_Ω is convex, then

$$c_k(X_\Omega) = \min \left\{ \|v\|_\Omega \;\middle|\; v \in (\mathbb{N} \cup \{0\})^n, \ \sum v_j = k \right\}$$

where $\|v\|_\Omega = \max \{ \langle v, w \rangle \mid w \in \Omega \}$

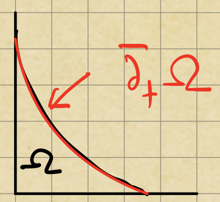
- computing c_k involves comparison of $\binom{k+n-1}{n-1}$ (similar) optimization problems

II) If X_Ω is concave, then

$$c_k(X_\Omega) = \max\left\{ [v]_\Omega \;\middle|\; v \in \mathbb{N}^n, \sum v_j = k+n-1 \right\}$$

where $[v]_\Omega = \min\left\{ \langle v, w\rangle \mid w \in \overline{\{\partial\Omega \cap \mathbb{R}^n_{>0}\}} = \bar{\partial}_+\Omega \right\}$

i.e



Capacities and the Minkowski sum

Theorem (Artstein-Avidan, Ostrover)

If U and V are convex bodies in $\mathbb{R}^{2n}$, then

$$(c_1(U+V))^{1/2} \geq (c_1(U))^{1/2} + (c_1(V))^{1/2}$$

with equality iff ∂U and ∂V have homothetic representatives of c_1.

Q1 Does this inequality hold for c_k with $k>1$?

Al No.

Theorem (K., Liang)

For even k

$$c_k\left(E\left(\left(1+\tfrac{1}{k}\right)^2, 1\right) + E\left(1, \left(1+\tfrac{1}{k}\right)^2\right)\right)^{1/2} < c_k\left(E\left(\left(1+\tfrac{1}{k}\right)^2, 1\right)\right)^{\frac{1}{2}} + c_k\left(E\left(1, \left(1+\tfrac{1}{k}\right)^2\right)\right)^{\frac{1}{2}}$$

For odd k >1

$$c_k\left(E(1,1) + E\left(\left(1-\tfrac{1}{k}\right)^2, 1\right)\right)^{1/2} < c_k(E(1,1))^{1/2} + c_k\left(E\left(\left(1-\tfrac{1}{k}\right)^2, 1\right)\right)^{\frac{1}{2}}$$

- $c_k\left(E(a,b) + E(c,d)\right)$ can be made explicit

Observation : Ostrover

<u>Prop</u> (A-A, O) If a (normalized) capacity C satisfies the symplectic Brunn-Minkowski inequality then for every centrally symmetric convex body U

$$C(U) \leq \pi \left(\frac{\text{mean-width}(U)}{2} \right)^2$$

Applying to C_k and $U = P(1,1)$ implies $C_{k>1}$ do not satisfy symplectic Brunn-Minkowski for $k \neq 3, 5, 7$.

Steiner Formula for $U \subset \mathbb{R}^m$ convex

$$\mathrm{Vol}\left(U + t\, B^m(1) \right) = \sum_{j=0}^{m} \binom{m}{j} W_j(U)\, t^j$$

$W_j(U) = j^{th}$ Quermassintegrale of U

$$W_{m-1}(U) = \frac{\mathrm{Vol}(B^m(1))}{2} \left(\text{mean-width}\,(U) \right)$$

Q Are there symplectic Steiner formulas?

$$c_k\left(U + t\, B^{2n}(1) \right) \overset{?}{=} c_k(U) + a_k(U)\, t + c_k(B^2(1))\, t^2$$

$a_k(U)$ = "k^{th} symplectic mean width"?

Artstein-Avidan, Ostrover $\Rightarrow$ $a_1(U) \geq 2\sqrt{\pi}\sqrt{c_1(U)}$

with equality iff $c_1(U)$ is represented by a great circle on ∂U.

<u>a</u> Not in the form above. For $a > \sqrt{2}$

$$c_2(E(\pi,\pi a^2)+tB) = \begin{cases} 2\pi + 4\pi t + 2\pi t^2, & t \leq \dfrac{a-\sqrt{2}}{\sqrt{2}-1} \\ \pi a^2 + 2\pi a + \pi t^2, & t > \dfrac{a-\sqrt{2}}{\sqrt{2}-1} \end{cases}$$

Relation of capacities to volume

- The $c_k(E(1,a)) = \text{Sort}\{\mathbb{Z} \cup a\mathbb{Z}\}[k]$ "see" $\text{Vol}(E(1,a)) = \frac{a}{2}$.

- For $P(1,a)$ the $c_k(P(1,a)) = k$ are completely blind to $\text{Vol}(P(1,a)) = a$.

<u>Q2</u> How do these blind spots develop?

Consider $E_p(1,a) = \left\{ (\pi|z_1|^2)^p + \left(\frac{\pi|z_2|^2}{a}\right)^p \leq 1 \right\}$

which go from $E(1,a)$ at $p=1$ to $P(1,a)$ as $p \to \infty$.

Study $c_k(E_p(1,a))$

Lemma 1 (K.L.) For each k $\exists$ $p(k) < \infty$ s.t.

$$c_k(E_p(1,a)) = k = c_k(P(1,a)) \quad \forall \quad p \geq p(k)$$

$p \mapsto c_k(E_p(1,a))$

Lemma 2 (K. L.) For each p $\exists$ $k(p)$ s.t.

$$\frac{d}{da}\left(c_k(E_p(1,a)) \right) > 0 \quad \forall \; k > k(p)$$

$$\left(\begin{array}{l} c_k(E_p(1,a)) = \left((a(k-m))^{\frac{p}{p-1}} + m^{\frac{p}{p-1}} \right)^{\frac{p-1}{p}} \\ \text{for some } m \in [1, k-1] \end{array} \right)$$

so each $c_k(E_p(1,a))$ with $k > k(p)$ "sees" a

and hence $Vol(E_p(1,a))$

A2 $\{ c_k(E_p(1,a)) \}_{k \in \mathbb{N}}$ is only blind to $\mathrm{Vol}(E_p(1,a))$ in the limit $p \to \infty$.

Q3 If ∂X is smooth does $\{ c_k(X) \}_{k \in \mathbb{N}}$ "see" $\mathrm{Vol}(X)$?

A3 No.

Symmetry and simplification.

Def^{n} $\Omega \subset \mathbb{R}^n_{\geq 0}$ is symmetric if

$(x_1, \dots, x_n) \in \Omega \Rightarrow (x_{\sigma(1)}, \dots, x_{\sigma(n)}) \in \Omega \quad \forall \sigma \in S_n.$

Theorem (K., Liang)

I) If Ω is symmetric and X_Ω is convex, then

$$c_k(X_\Omega) = \max_{w \in \Omega} \langle w, V(k,n) \rangle = \| V(k,n) \|_\Omega \quad \text{for}$$

$$V(k,n) = \Big(\left\lfloor \tfrac{k}{n} \right\rfloor, \dots, \left\lfloor \tfrac{k}{n} \right\rfloor, \underbrace{\left\lceil \tfrac{k}{n} \right\rceil \quad \left\lceil \tfrac{k}{n} \right\rceil}_{k \bmod n} \Big).$$

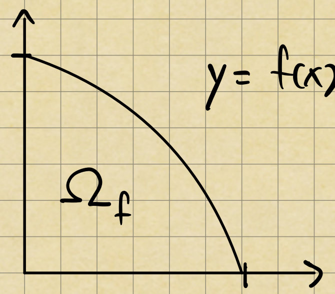
II) If Ω is symmetric and X_Ω is concave,

then $$c_k(X_\Omega) = \min_{w \in \overline{\partial_+ \Omega}} \langle \hat{V}(k,n), w \rangle = [\hat{V}(k,n)]_\Omega$$

where

$$\hat{V}(k,n) = \Big(\underbrace{\left\lceil \frac{k+n-1}{n} \right\rceil, \ldots, \left\lceil \frac{k+n-1}{n} \right\rceil}_{k+n-1 \bmod n}, \left\lfloor \frac{k+n-1}{n} \right\rfloor, \ldots, \left\lfloor \frac{k+n-1}{n} \right\rfloor \Big)$$

Example



f1) $f^{-1} = f \Rightarrow \Omega_f$ symmetric

f2) $f'' < 0$

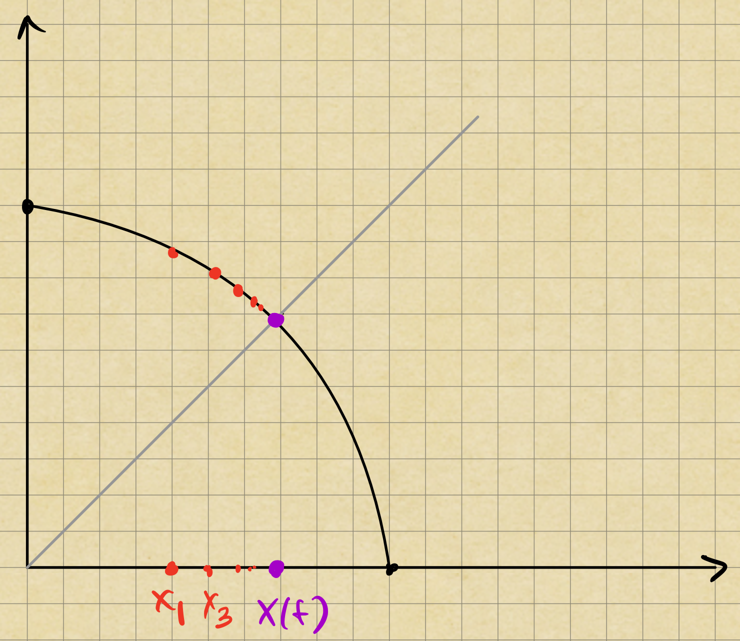
f3) $f'(0) \in (-\frac{1}{2}, 0)$

$f(x(f)) = x(f)$ $\qquad$ $f'(x_k) = -\frac{k-1}{k+1}$, k odd

$$C_k(X_\Omega) = \begin{cases} k\, x(f), & k \text{ even} \\ \frac{k-1}{2} x_k + \frac{k+1}{2} f(x_k), & k \text{ odd} \end{cases}$$

The c_k are all determined by

$x_k \nearrow x(f)$ and $f(x_k)$.



Perturbations of f away from $x_k \nearrow x(f)$ can change the volume while keeping the capacities fixed.

$f_\delta = f + \delta\Big($ 

$\Big)$ + mirror bump

$|\delta|$ suff small $\Rightarrow$ f_δ satisfies (f1)–(f3)

$$c_k(X_{\Omega_{f_\delta}}) = c_k(X_{\Omega_f}) \quad \forall\, k \in \mathbb{N}$$

$$\mathrm{Vol}(X_{\Omega_{f_\delta}}) \neq \mathrm{Vol}(X_{\Omega_f})$$

The $\{c_k\}_{k\in\mathbb{N}}$ do not see Vol.

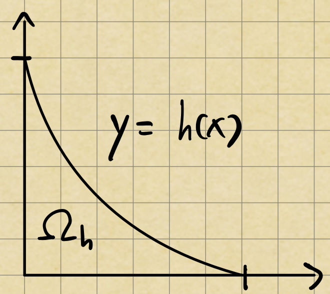
q Given $U \subset \mathbb{R}^{2n}$ convex with ∂U smooth,

is $\sup \dfrac{Vol(V)}{Vol(W)} < \infty$ where

$c_k(V) = c_k(W) = c_k(U) \quad \forall \quad k \in \mathbb{N}.$

Without convexity, the answer is No!

In the concave toric setting, consider



h1) $h^{-1} = h$

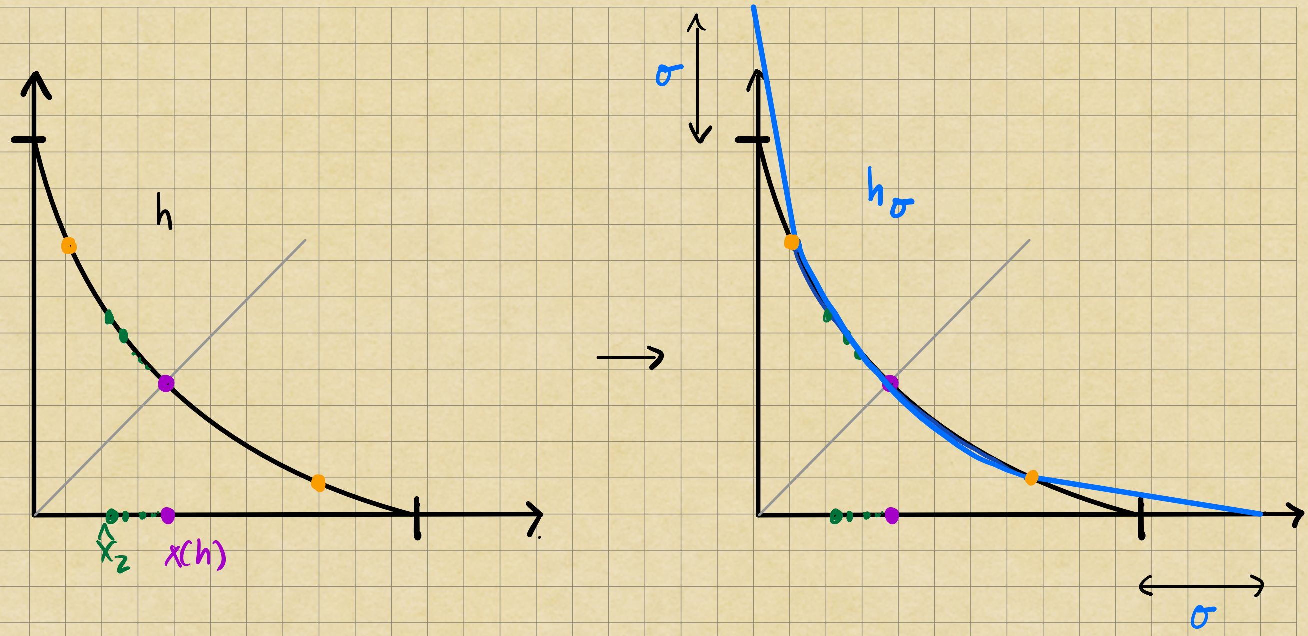
h2) $h'' > 0$

h3) $h'(0) \in (-\infty, -2)$

Thm (K. L.) implies

$$c_k(X_{\Omega_h}) = \begin{cases} (k+1)\, x(h) & \text{for } k \text{ odd} \\ \frac{k+2}{2}\hat{x}_k + \frac{k}{2} h(\hat{x}_k) & k \text{ even} \end{cases}$$

where $h'(\hat{x}_k) = -\frac{k+2}{k}$.



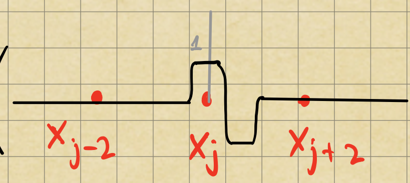
$$c_k(X_{\Omega_{h_\sigma}}) = c_k(X_{\Omega_h})$$

$$\mathrm{Vol}(X_{\Omega_{h_\sigma}}) \longrightarrow \infty \quad \text{as} \quad \sigma \to \infty$$

Q4 Are the c_k independent ?

A4 Yes

For $j = 2n+1$ consider f as above and form

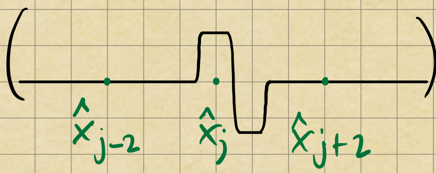
$f_\delta = f + \delta \Big($  $\Big)$ + mirror bump

$$c_k(X_{\Omega_{f_\delta}}) = c_k(X_{\Omega_f}) \quad \text{for} \quad k \neq j$$

$$c_j(X_{\Omega_{f_\delta}}) > c_j(X_{\Omega_f}) \quad \left(= c_j(X_{\Omega_f}) + \left(\tfrac{j+1}{2}\right)\delta \right)$$

(Rmk Volume is independent of δ.)

For $j = 2n$ use a similar trick in concave setting

$h_\delta = h + \delta \Big($ 

$\Big)$ + mirror bump

q $\exists ?$ X such that for all $j \in \mathbb{N}$

there is a Y_j such that

$c_k(Y_j) = c_k(X) \quad \forall\, k \neq j$ and

$c_j(Y_j) \neq c_j(X)$.

Q5: If $U \subset \mathbb{R}^4$ has smooth boundary and is strictly convex, do the $c_k(U)$ and $Vol(U)$ determine U up to symplectomorphism?

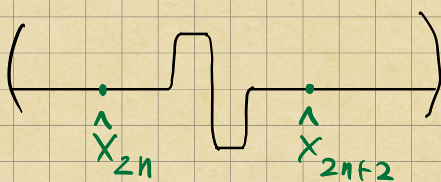
A5: No.

Tools: ECH capacities of Hutchings, c_k^{ECH}, + algorithm for $c_k^{ECH}(X_{\Omega_h})$ developed by

Choi, Christofaro-Gardiner, Frenkel, Hutchings, Ramos

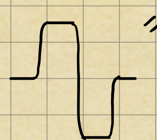
Strategy

- Consider $\mathcal{U} = X_{\Omega_h}$ as above.
- Deform h away from $\hat{x}_k \nearrow x(h)$ as

$h_\delta = h + \delta\Big($ 

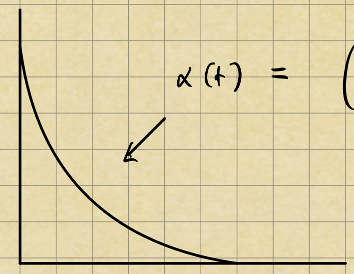
$\Big)$ + mirror bump

<u>Note</u> the c_k and Vol are unchanged!

- Show "  " $\Rightarrow$ $c_\ell^{ECH}(X_{\Omega_{h_\delta}}) \neq c_\ell^{ECH}(X_{\Omega_h})$

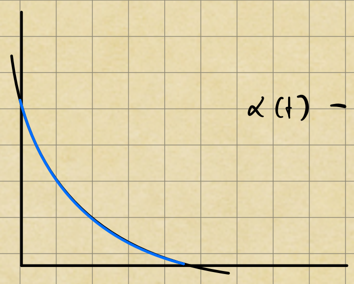
for some ℓ.

Example from K. L



$\alpha(t) = \left(2\sin\left(\frac{t}{2}\right) - t\cos\left(\frac{t}{2}\right),\ 2\sin\left(\frac{t}{2}\right) + (2\pi - t)\cos\left(\frac{t}{2}\right)\right)$

(Ramos) $\quad t \in [0, 2\pi]$

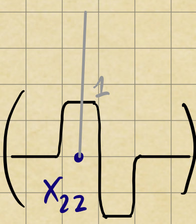


$\alpha(t) - (\varepsilon, \varepsilon)$ = graph of h

- The $c_k^{ECH}(X_{\Omega_h})$ depend on h at points

$x_1,\ x_2,\ x_{11},\ x_{12},\ x_{21},\ x_{22},\ \dots$

- X_{22} lies away from $\hat{x}_k \nearrow x(h)$

- For $h_\delta = h + \delta\Big($  $\Big)$ + mirror bump

$$c_q^{ECH}(X_{\Omega_{h_\delta}}) = c_q^{ECH}(X_{\Omega_h}) + \delta$$

for all suff small $\delta > 0$.

Yuanpu's Proof of the simplified formulas

Given Ω symmetric s.t. X_Ω is convex. Need

$$c_k(X_\Omega) \overset{G-H}{=} \min\left\{ \|v\|_\Omega \,\middle|\, v \in (\mathbb{N} \cup \{0\})^n,\ \sum v_j = k \right\}$$

$$\overset{L-K}{=} \|V(k,n)\|_\Omega$$

for $V(k,n) = \Big(\underbrace{\left\lfloor \tfrac{k}{n} \right\rfloor, \ldots, \left\lfloor \tfrac{k}{n} \right\rfloor}, \overbrace{\left\lceil \tfrac{k}{n} \right\rceil, \ldots, \left\lceil \tfrac{k}{n} \right\rceil}^{k \bmod n} \Big)$

and $\|v\|_\Omega = \max\limits_{w \in \Omega} \langle v, w \rangle$.

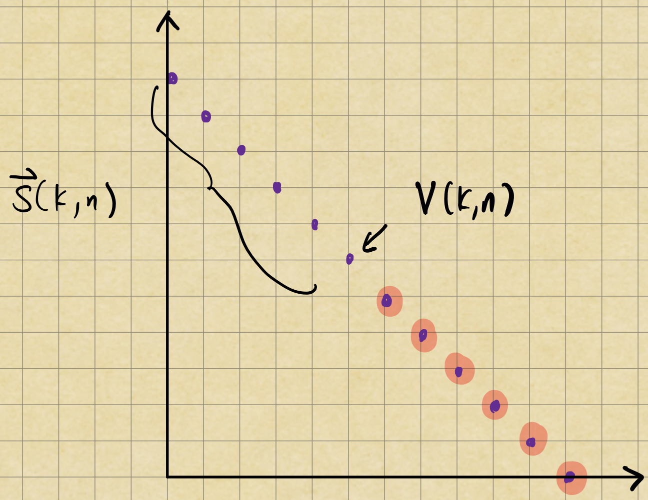
$\underline{\text{Def}^{n}}$ $v \in \mathbb{R}^n$ is ordered if $v_1 \leq v_2 \leq \ldots \leq v_n$

eg $V(k,n)$ is ordered.

Symmetry of Ω $\Rightarrow$ $\| (v_1, \ldots, v_n) \|_\Omega = \| (v_{\sigma(1)}, \ldots, v_{\sigma(n)}) \|_\Omega$
$\forall \ \sigma \in S_n$

$$\vec{S}(k,n) = \left\{ v \in (\mathbb{N} \cup \{0\})^n \;\middle|\; \sum v_j = k , \ v \text{ ordered} \right\}$$

$$c_k(X_\Omega) = \min \left\{ \|v\|_\Omega \;\middle|\; v \in \vec{S}(k,n) \right\}$$



$\vec{S}(k,n)$
$V(k,n)$

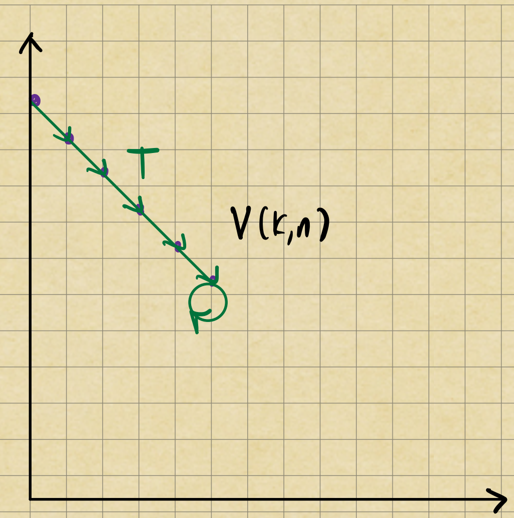
Consider the map $T : \vec{S}(k,n) \to \vec{S}(k,n)$

$$V = (\underbrace{v_1, v_1, \ldots, v_1}_{m}, \ldots, \underbrace{v_n, \ldots, v_n}_{M})$$

$$\Big\downarrow T$$

$$\begin{cases} (\underbrace{v_1, \ldots, v_1}_{m-1}, v_1+1, \ldots, v_n-1, \underbrace{v_n \cdots v_n}_{M-1}) & \text{if } v_n > v_1+1 \\ V & \text{otherwise} \end{cases}$$

- $\mathrm{Fix}(T) = \{V(k,n)\}$ and $T^j(V) = V(k,n)$ for $j \gg 1$



Prop $\|T(v)\|_{\Omega} \leq \|v\|_{\Omega}$

this settles things

__lemma__ $\| v \|_{\Omega} = \langle v, w \rangle$ for an ordered $w \in \Omega$

Pf Assume $\| v \|_{\Omega} = \langle v, w \rangle$ and $w_j > w_{j+\ell}$

Set $\tilde{w} = (w_1 , \dots , w_{j+\ell} , \dots , w_j \dots w_n)$

$$\langle v, \tilde{w} \rangle - \langle v, w \rangle = v_j w_{j+\ell} + v_{j+\ell} w_j - v_j w_j - v_{j+\ell} w_{j+\ell}$$

$$= \underbrace{(v_j - v_{j+\ell})}_{-ve} \underbrace{(w_{j+\ell} - w_j)}_{-ve.}$$

$$\geq 0$$

Since $\langle v, w \rangle = \max_{w \in \Omega} \langle v, w \rangle$ we have $\langle v, \tilde{w} \rangle = \langle v, w \rangle$

Proof of Prop : $\| D(v) \|_{\Omega} \leq \| v \|_{\Omega}$

lemma $\Rightarrow$ $\| T(v) \|_{\Omega} = \langle T(v), w \rangle$ for w ordered

$$\| v \|_{\Omega} - \| T(v) \|_{\Omega} \geq \langle v, w \rangle - \langle T(v), w \rangle$$

$$= (v_1 w_t + v_n w_{n-T}) - (v_1 + 1) w_t - (v_n - 1) w_{n-T}$$

$$= w_{n-T} - w_t$$

≥ 0 since w is ordered.