Topological entropy and Floer theory
joint work with Viktor Ginzburg and Başak Gürel

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February 25, 2022
Introduction

How much does Floer theory know about dynamics beyond periodic orbits?

Here one can look at different dynamics features. In this work we focus on “topological entropy”.

**Question:** Can one detect the topological entropy of a (compactly supported) Hamiltonian diffeomorphism using the data coming from Floer theory?
Topological entropy

Let \((X, d)\) be a compact metric space and \(f : X \to X\) be a continuous map. For \(k \in \mathbb{N}\), define

\[
d_k(x, y) := \max_{0 \leq i \leq k-1} \{d(f^i(x), f^i(y))\}.
\]
Topological entropy

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\]

For \(\epsilon > 0\), let \(S_{\epsilon}(k)\) be the maximal number of \(\epsilon\)-separated points with respect to the metric \(d_k\). The topological entropy of \(f\) is

\[
h_{\text{top}}(f) := \lim_{\epsilon \downarrow 0} \lim_{k \to \infty} \sup \frac{\log S_{\epsilon}(k)}{k}.
\]

Example: Let \(f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}; x \to 2x\). Then \(h_{\text{top}}(f) = \log 2 = 1\).
Topological entropy

More examples:

(i) On closed surfaces, $h_{\text{top}}(\varphi) = 0$ for autonomous Hamiltonian diffeomorphisms $\varphi$ (not true in higher dimensions).

(ii) Katok (1980): For $C^{1+\epsilon}$ diffeomorphisms $\varphi$ of closed surfaces

$$h_{\text{top}}(\varphi) \leq \limsup_{k \to \infty} \frac{\log |\text{Fix} \varphi^k|}{k}.$$ 

(iii) In higher dimensions, (even) a symplectomorphism $\varphi$ without periodic points can have $h_{\text{top}}(\varphi) > 0$. 
Barcode entropy

Setting:
- \((M, \omega)\) closed monotone symplectic manifold
- \(L \subset M\) closed monotone Lagrangian with minimal Chern number \(N_L \geq 2\)
- \(\varphi \in \text{Ham}(M, \omega)\)

Remark: All maps and manifolds are \(C^\infty\).
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\[ b_\epsilon(L, \varphi(L)) := |\{\text{bars of length} > \epsilon \text{ in } CF(L, \varphi(L))\}|. \]
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Remark: All maps and manifolds are \(C^\infty\).

Let

\[ b_\epsilon(L, \varphi(L)) := |\{\text{bars of length} > \epsilon \text{ in } \text{CF}(L, \varphi(L))\}|. \]

Note: We use \(b_\epsilon(L, \varphi(L))\) as a lower bound for the number of Hofer-stable intersections. Namely, if \(d_H(\tilde{L}, L) < \delta < \epsilon/2\) and \(\tilde{L} \pitchfork \varphi(L)\), then

\[ b_\epsilon(L, \varphi(L)) \leq b_{\epsilon-2\delta}(\tilde{L}, \varphi(L)) \leq |\tilde{L} \cap \varphi(L)|. \]
Note (recall): If $d_H(\tilde{L}, L) < \delta < \epsilon/2$ and $\tilde{L} \cap \varphi^k(L)$, then

$$b_\epsilon(L, \varphi^k(L)) \leq |\tilde{L} \cap \varphi^k(L)|.$$
Note (recall): If \( d_H(\tilde{L}, L) < \delta < \epsilon/2 \) and \( \tilde{L} \cap \varphi^k(L) \), then

\[
b_{\epsilon}(L, \varphi^k(L)) \leq |\tilde{L} \cap \varphi^k(L)|.\]

**Definition (Relative barcode entropy)**

The \( \epsilon \)-barcode entropy of \( \varphi \) relative to \( L \) is

\[
\bar{h}_\epsilon(\varphi; L) := \limsup_{k \to \infty} \frac{\log^+ b_{\epsilon}(L, \varphi^k(L))}{k}
\]

and the barcode entropy of \( \varphi \) relative to \( L \) is

\[
\bar{h}(\varphi; L) := \lim_{\epsilon \downarrow 0} \bar{h}_\epsilon(\varphi, L) \in [0, \infty].
\]
Barcode entropy

As in the Lagrangian case, let

\[ b_\epsilon(\varphi) := |\{ \text{bars of length } > \epsilon \text{ in } \text{CF}(\varphi) \}|. \]

**Remark:** In \( \text{CF}(\varphi) \) we work with all free homotopy classes of loops.
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Remark: In $CF(\varphi)$ we work with all free homotopy classes of loops.

**Definition (Absolute barcode entropy)**

The barcode entropy of $\varphi$ is

\[ h(\varphi) := \lim_{\epsilon \searrow 0} \limsup_{k \to \infty} \frac{\log^+ b_\epsilon(\varphi^k)}{k} \in [0, \infty]. \]

Note: $h(\varphi) = h(id \times \varphi; \Delta)$ where $\Delta \subset M^- \times M$ is the diagonal.
Barcode entropy

Some formal properties:

(i) $\bar{h}(\varphi) = \bar{h}(\varphi^{-1})$ and $\bar{h}(\varphi) = \bar{h}(\psi\varphi\psi^{-1})$.
(ii) $\bar{h}(\varphi^k) \leq k \bar{h}(\varphi)$.
(iii) $\bar{h}(\varphi \times \psi) \leq \bar{h}(\varphi) + \bar{h}(\psi)$.
(iv) $\bar{h}(\varphi; L)$ is lower semi-continuous in $L$ with respect to the Hofer metric.

Topological entropy:

(i) $h_{\text{top}}(\varphi) = h_{\text{top}}(\varphi^{-1})$ and $h_{\text{top}}(\varphi) = h_{\text{top}}(\psi\varphi\psi^{-1})$.
(ii) $h_{\text{top}}(\varphi^k) = k h_{\text{top}}(\varphi)$.
(iii) $h_{\text{top}}(\varphi \times \psi) = h_{\text{top}}(\varphi) + h_{\text{top}}(\psi)$.
Main results

**Theorem A**

Let $L$ be a closed monotone Lagrangian submanifold with minimal Chern number $N_L \geq 2$ in a symplectic manifold $M$ and let $\varphi: M \to M$ be a compactly supported Hamiltonian diffeomorphism. Then

$$\bar{h}(\varphi; L) \leq h_{\text{top}}(\varphi).$$

As a consequence, $\bar{h}(\varphi; L) < \infty$. 
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$$\bar{h}(\varphi; L) \leq h_{\text{top}}(\varphi).$$

As a consequence, $\bar{h}(\varphi; L) < \infty$. Since $\bar{h}(id \times \varphi; \Delta) = \bar{h}(\varphi)$ and $h_{\text{top}}(id \times \varphi) = h_{\text{top}}(\varphi)$, we have:

**Corollary A**

Let $\varphi : M \to M$ be a Hamiltonian diffeomorphism of a closed monotone symplectic manifold $M$. Then

$$\bar{h}(\varphi) \leq h_{\text{top}}(\varphi).$$
Main results

A compact invariant (hyperbolic) set $K$ of $\varphi$ is called *locally maximal* if there exists a neighborhood $U \supset K$ such that $K = \bigcap_{k \in \mathbb{Z}} \varphi^k(U)$. 
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**Theorem B**

Let $\varphi: M \rightarrow M$ be a Hamiltonian diffeomorphism of a closed monotone symplectic manifold $M$ and let $K \subset M$ be a locally maximal hyperbolic subset. Then

$$\bar{h}(\varphi) \geq h_{\text{top}}(\varphi|_K).$$

**Example:** Smale’s horseshoe is a locally maximal hyperbolic set.

**Katok-Hasselblatt:** $h_{\text{top}}(\varphi|_K) = \limsup_{k \rightarrow \infty} \frac{\log |\text{Fix} \varphi^k|_K|}{k}$ where $K$ is locally maximal and hyperbolic.
Katok (1980): On closed surfaces

\[ h_{\text{top}}(\varphi) = \sup \{ h_{\text{top}}(\varphi|K) \mid K \text{ is locally maximal and hyperbolic} \}. \]
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**Theorem C**

Let \( \varphi: M \rightarrow M \) be a Hamiltonian diffeomorphism of a closed surface \( M \). Then

\[ \bar{h}(\varphi) = h_{\text{top}}(\varphi). \]

Proof of Theorem C: Corollary A + Theorem B + Katok's result.

Corollary A: \( \bar{h}(\varphi) \leq h_{\text{top}}(\varphi). \)

Theorem B: \( \bar{h}(\varphi) \geq h_{\text{top}}(\varphi|_K). \)
Main results

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**Theorem C**

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**Proof of Theorem C:** Corollary A + Theorem B + Katok's result.

**Corollary A:** \( \bar{h}(\varphi) \leq h_{\text{top}}(\varphi). \)

**Theorem B:** \( \bar{h}(\varphi) \geq h_{\text{top}}(\varphi|_K). \)

**Remark:** In Theorem B (and C), it is essential that the Floer complex is generated by all fixed points (or all intersections in the Lagrangian case).
Previous works

Reeb flows and symplectomorphisms:


Hamiltonian diffeomorphisms in dimension 2:

Humilière (2017): Let $L \subset S^2$ be an equator. There exists $C > 0$ such that for all $\varphi \in \text{Ham}(S^2)$, $\limsup_{k \to \infty} d_H(L, \varphi^k(L)) / k \leq C h_{\text{top}}(\varphi)$.

Khanevsky (2021): Let $L \subset \Sigma_{g \geq 1}$ be simple and non-contractible. For all $N > 0$, there exists $L_N \subset \Sigma_{g \geq 1}$ such that every $\varphi \in \text{Ham}(\Sigma_{g \geq 1})$ with $\varphi(L) = L_N$ has $h_{\text{top}}(\varphi) \geq N$. 
Previous works

Chor-Meiwes (2021): For all $N > 0$, there exists open dense subset $U$ of $(\text{Ham}(\Sigma_{g \geq 2}), d_H)$ such that $h_{\text{top}}|_U \geq N$.

Alves-Meiwes (2021): The topological entropy $h_{\text{top}} : (\text{Ham}(\Sigma), d_H) \to \mathbb{R}$ is lower semi-continuous.

Note: By Theorem C, all of these results hold for the absolute barcode entropy as well. We cannot prove directly any of them at the moment.
Barcodes

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Setup: Suppose that $L \pitchfork \varphi(L)$. The Lagrangian Floer complex

$$(\text{CF}(L, \varphi(L)), d_{Fl})$$

is generated by all (capped) intersections $L \cap \varphi(L)$ over the universal Novikov field $\Lambda_{\mathbb{F}_2}$ and filtered by the action $\mathcal{A}$ (the Floer differential $d_{Fl}$ strictly decreases the action $\mathcal{A}$).
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Note that both $d_{Fl}$ and $\mathcal{A}$ depend on other data (the Hamiltonian $H$ generating $\varphi$, the complex structure $J$ and the choice of cappings). We implicitly make these choices.
Usher-Zhang: A basis $\Sigma = \{\alpha_i, \gamma_j, \eta_j\}$ of $\text{CF}(L, \varphi(L))$ over $\Lambda_{\mathbb{F}_2}$ is called a singular value decomposition if

$-d_{FL}(\alpha_i) = 0$ and $d_{FL}(\gamma_j) = \eta_j$,

- $\Sigma$ is orthogonal (in the non-Archimedean sense).
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Note: A subset $\{\beta_i\} \subset \text{CF}(L, \varphi(L))$ is called orthogonal, if

$$\mathcal{A}(\sum \lambda_i \beta_i) = \max \mathcal{A}(\lambda_i \beta_i)$$

for all $\lambda_i \in \Lambda_{\mathbb{F}_2}$. For example, suppose that $\Lambda_{\mathbb{F}_2} = \mathbb{F}_2$ and all capped intersections have distinct actions, then $\{\beta_i\}$ is orthogonal if and only if $\mathcal{A}(\beta_i)$ are distinct. Similarly, if $\mathcal{A}(\beta_1) < \mathcal{A}(\beta_2)$, then $\{\beta_2, \beta_1 + \beta_2\}$ is not orthogonal. Roughly speaking, we don’t want “cancellations”.
The barcode $\mathcal{B}$ of $(\text{CF}(L, \varphi(L), d_{Fl}))$ is the multiset formed by the finite bars $A(\gamma_j) - A(\eta_j)$ together with $\dim_{\mathbb{F}_2} \text{HF}(L, \varphi(L))$ many $\infty$-bars.
The barcode $\mathcal{B}$ of $(\text{CF}(L, \varphi(L), d_{F^I})$ is the multiset formed by the finite bars $A(\gamma_j) - A(\eta_j)$ together with $\dim_{\mathbb{F}_2} \text{HF}(L, \varphi(L))$ many $\infty$-bars.

Usher-Zhang:

(i) $(\text{CF}(L, \varphi(L), d_{F^I})$ admits a singular value decomposition.

(ii) Its barcode $\mathcal{B}$ only depends on $\varphi$ and $L$.

(iii) If $d_H(\tilde{L}, L) < \delta < \epsilon/2$, then $b_\epsilon(L, \varphi(L)) \leq b_{\epsilon-2\delta}(\tilde{L}, \varphi(L))$. 
The barcode $B$ of $(\text{CF}(L, \varphi(L), d_{Fl})$ is the multiset formed by the finite bars $A(\gamma_j) - A(\eta_j)$ together with $\dim_{\Lambda F_2} \text{HF}(L, \varphi(L))$ many $\infty$-bars.

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Note:

(i) If $L \cap \varphi(L)$, then $b_\epsilon(L, \varphi(L)) \leq |L \cap \varphi(L)|$.
(ii) Since $\dim_{\Lambda F_2} \text{HF}(L, \varphi(L))$ does not depend on $\varphi$, if $\tilde{h}(\varphi, L) \neq 0$, then the growth comes from finite bars.
Proof of Theorem A

**Theorem A:** \( \bar{h}(\varphi; L) \leq h_{\text{top}}(\varphi). \)
Proof of Theorem A

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Yomdin’s theorem: Let $\varphi : M \to M$ be a $C^\infty$-diffeomorphism and $N \subset M$ be a compact submanifold. Then

$$\limsup_{k \to \infty} \frac{\log \text{vol}(\varphi^k(N))}{k} \leq h_{\text{top}}(\varphi).$$
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Theorem A: $\hat{h}(\varphi; L) \leq h_{\text{top}}(\varphi)$.

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**Proof of Theorem A:** Suppose that $\hat{h}(\varphi; L) \neq 0$ and let $0 < \alpha < \hat{h}(\varphi; L)$.

**Step 1:** Set $L^k := \varphi^k(L)$. Let $\epsilon > 0$ and $k_i \to \infty$ such that

$$\text{const } 2^{k_i \alpha} \leq b_\epsilon(L, L^{k_i}).$$

By Yomdin’s theorem, it suffices to show that

$$\text{const } 2^{k_i \alpha} \leq \text{vol}(L^{k_i}).$$
Step 2: (Crofton’s inequality)

Let $N \subset M$ be a closed submanifold and let $B$ be a compact manifold.

Let $\Psi : B \times N \to M$ be a submersion (onto its image) with $\Psi|_{s \times N}$ an embedding for all $s \in B$. Set $N_s := \Psi(s, N)$. 
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Lemma

Let $\tilde{N} \subset M$ be a closed submanifold with $\text{codim} \tilde{N} = \dim N$. We have

$$\int_B |N_s \cap \tilde{N}| \, ds \leq \text{const} \, \text{vol}(\tilde{N}).$$

where the constant does not depend on $\tilde{N}$. 
Proof of Theorem A:

Step 3: (Lagrangian tomograph)

Let \( \Psi : B^d \times L \to T^*L \subset M \) be a submersion (onto its image) with

(i) \( L_s := \Psi(s, L) \) is an embedded Lagrangian for all \( s \in B^d \),

(ii) \( d_H(L_s, L) < \delta < \epsilon/2 \) for all \( s \in B^d \).

Remark: Lagrangian tomographs exist (with \( d = 2 \dim L - 1 \)).
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Remark: Lagrangian tomographs exist (with $d = 2 \dim L - 1$).

By Hofer stability of $b_\epsilon(L, L^{k_i})$, we have

$$\text{const}\ 2^{k_i \alpha} \leq b_\epsilon(L, L^{k_i}) \leq b_{\epsilon-2\delta}(L_s, L^{k_i}) \leq |L_s \cap L^{k_i}|$$

for almost all $s \in B^d$. 
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**Remark:** Lagrangian tomographs exist (with \( d = 2\dim L - 1 \)).

By Hofer stability of \( b_\epsilon(L, L^{k_i}) \), we have

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\text{const } 2^{k_i\alpha} \leq b_\epsilon(L, L^{k_i}) \leq b_{\epsilon - 2\delta}(L_s, L^{k_i}) \leq |L_s \cap L^{k_i}|
\]

for almost all \( s \in B^d \). Then, by Crofton’s inequality, we have

\[
\text{const } 2^{k_i\alpha} \leq \text{vol}(L^{k_i}) \implies \alpha \leq h_{\text{top}}(\varphi).
\]
Theorem B: $\bar{h}(\varphi) \geq h_{\text{top}}(\varphi|_K)$ where $K \subset M$ is locally maximal and hyperbolic.
Proof of Theorem B

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**Proof of Theorem B:**

Step 1: We say that a fixed point \( x \in \text{Fix}(\varphi) \) is \( \epsilon \)-isolated if all Floer cylinders \( u_x \) asymptotic to \( x \) has energy \( E(u_x) > \epsilon \).

**Lemma**

Suppose that \( \varphi \) has \( p \) \( \epsilon \)-isolated fixed points. Then \( b_\epsilon(\varphi) \geq p/2 \).
Proof of Theorem B

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Lemma

Suppose that \( \varphi \) has \( p \) \( \epsilon\)-isolated fixed points. Then \( b_\epsilon(\varphi) \geq p/2 \).

Katok-Hasselblatt: \( h_{\text{top}}(\varphi|_K) = \limsup_{k \to \infty} \frac{\log |\text{Fix} \varphi^k|_K|}{k} \).

It suffices to show that there exists \( \epsilon_K > 0 \) such that all periodic points contained in \( K \) are \( \epsilon_K\)-isolated.
Step 2:

Ginzburg-Gürel (2018): (Crossing Energy Theorem)

Let $\tilde{K} \subset \tilde{U} \subset S^1 \times M$ be an isolating neighborhood. There exist $\epsilon > 0$ such that all Floer cylinders $u$ that are

(i) asymptotic to a $k$-periodic point contained in $K$,

(ii) $\tilde{u} \not\subset \tilde{U}$

have energy $E(u) > \epsilon$.

Remark: Here $\epsilon > 0$ is independent of $k$. 
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have energy $E(u) > \epsilon$.

Remark: Here $\epsilon > 0$ is independent of $k$.

Next we focus on Floer cylinders that are contained in $\tilde{U}$. 
Proof of Theorem B

Step 3:

A set \( \{z_i \mid i \in \mathbb{Z}_k\} \subset M \) is called an \( \eta \)-pseudo-orbit of \( \varphi \), if

\[
d(\varphi(z_i), z_{i+1}) < \eta \quad \text{for all} \quad i \in \mathbb{Z}_k.
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Anosov Closing Lemma \( \implies \) There exists \( U \supset K \), \( \eta > 0 \) and \( \delta > 0 \) such that all \( \eta \)-pseudo-orbits contained in \( U \) are uniquely \( \delta \)-shadowed by a true orbit in \( K \).

Namely, if \( \{ z_i \mid i \in \mathbb{Z}_k \} \subset U \) is an \( \eta \)-pseudo-orbit, then there exists a unique periodic point \( x \in K \) of \( \varphi \) such

\[
d(\varphi^i(x), z_i) < \delta \quad \text{for all} \quad i \in \mathbb{Z}_k.
\]
Proof of Theorem B

Let \( u : \mathbb{R} \times S^1_k \rightarrow M \) be a Floer cylinder contained in \( \tilde{U} \). We have

\[
\{ u(\pm \infty, i) \mid i \in \mathbb{Z}_k \} \subset K.
\]

(i) \( E(u) \) is sufficiently small \( \implies \| \partial_s u \| \) is small point wise.

(ii) \( \| \partial_s u \| \) is small \( \implies \) there exists \( \eta > 0 \), independent of \( k \), such that

\[
\hat{z}(s) := \{ u(s, i) \mid i \in \mathbb{Z}_k \} \subset U \text{ is an } \eta\text{-pseudo-orbit for all } s \in \mathbb{R}.
\]
Proof of Theorem B

Let $u : \mathbb{R} \times S^1_k \to M$ be a Floer cylinder contained in $\tilde{U}$. We have

$$\{ u(\pm \infty, i) | i \in \mathbb{Z}_k \} \subset K.$$ 

(i) $E(u)$ is sufficiently small $\implies ||\partial_s u||$ is small point wise.

(ii) $||\partial_s u||$ is small $\implies$ there exists $\eta > 0$, independent of $k$, such that $\hat{z}(s) := \{ u(s, i) | i \in \mathbb{Z}_k \} \subset U$ is an $\eta$-pseudo-orbit for all $s \in \mathbb{R}$.

(iii) Anosov Closing Lemma $\implies$ if $\eta > 0$ is sufficiently small, then $\hat{z}(s)$ is shadowed by a unique periodic point $w(s) \in K$.

(iv) $w(s)$ depends continuously on $s$ $\implies u(-\infty, t) = u(+\infty, t)$. 
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(iv) $w(s)$ depends continuously on $s$ $\implies u(-\infty, t) = u(+\infty, t)$.

Conclusion: There exists $\epsilon_K > 0$ such that if $\tilde{u} \subset \tilde{U}$ and $E(u) \leq \epsilon_K$, then $u$ is constant.
The End