# Topological entropy and Floer theory
joint work with Viktor Ginzburg and Başak Gürel

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# Introduction

How much does Floer theory know about dynamics beyond periodic orbits?

Here one can look at different dynamics features. In this work we focus on "topological entropy".

Question: Can one detect the topological entropy of a (compactly supported) Hamiltonian diffeomorphism using the data coming from Floer theory?

# Topological entropy

Let $(X, d)$ be a compact metric space and $f : X \to X$ be a continuous map. For $k \in \mathbb{N}$, define

$$d_k(x, y) := \max_{0 \leq i \leq k-1} \{d(f^i(x), f^i(y))\}.$$

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For $\epsilon > 0$, let $S_\epsilon(k)$ be the maximal number of $\epsilon$-separated points with respect to the metric $d_k$. The *topological entropy of f* is

$$\mathrm{h}_{\mathrm{top}}(f) := \lim_{\epsilon \searrow 0} \limsup_{k \to \infty} \frac{\log S_\epsilon(k)}{k}.$$

Example: Let $f : \mathbb{R}/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$; $x \to 2x$. Then $\mathrm{h}_{\mathrm{top}}(f) = \log 2 = 1$.

# Topological entropy

More examples:

(i) On closed surfaces, $\mathrm{h}_{\mathrm{top}}(\varphi) = 0$ for autonomous Hamiltonian diffeomorphisms $\varphi$ (not true in higher dimensions).

(ii) Katok (1980): For $C^{1+\epsilon}$ diffeomorphisms $\varphi$ of closed surfaces

$$\mathrm{h}_{\mathrm{top}}(\varphi) \leq \limsup_{k \to \infty} \frac{\log |\operatorname{Fix} \varphi^k|}{k}.$$

(iii) In higher dimensions, (even) a symplectomorphism $\varphi$ without periodic points can have $\mathrm{h}_{\mathrm{top}}(\varphi) > 0$.

# Barcode entropy

Setting:

-$(M, \omega)$ closed monotone symplectic manifold
-$L \subset M$ closed monotone Lagrangian with minimal Chern number $N_L \geq 2$
-$\varphi \in Ham(M, \omega)$

Remark: All maps and manifolds are $C^\infty$.

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Note: We use $b_\epsilon(L, \varphi(L))$ as a lower bound for the number of Hofer-stable intersections. Namely, if $d_H(\tilde{L}, L) < \delta < \epsilon/2$ and $\tilde{L} \pitchfork \varphi(L)$, then

$$b_\epsilon(L, \varphi(L)) \leq b_{\epsilon - 2\delta}(\tilde{L}, \varphi(L)) \leq |\tilde{L} \cap \varphi(L)|.$$

# Barcode entropy

Note (recall): If $d_H(\tilde{L}, L) < \delta < \epsilon/2$ and $\tilde{L} \pitchfork \varphi^k(L)$, then

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**Definition (Relative barcode entropy)**

*The $\epsilon$-barcode entropy of $\varphi$ relative to $L$ is*

$$\hbar_\epsilon(\varphi; L) := \limsup_{k\to\infty} \frac{\log^+ b_\epsilon(L, \varphi^k(L))}{k}$$

*and the barcode entropy of $\varphi$ relative to $L$ is*

$$\hbar(\varphi; L) := \lim_{\epsilon \searrow 0} \hbar_\epsilon(\varphi, L) \in [0, \infty].$$

# Barcode entropy

As in the Lagrangian case, let

$$b_\epsilon(\varphi) := |\{\text{bars of length} > \epsilon \text{ in } \mathrm{CF}(\varphi)\}|.$$

Remark: In $\mathrm{CF}(\varphi)$ we work with all free homotopy classes of loops.

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**Definition (Absolute barcode entropy)**

*The barcode entropy of $\varphi$ is*

$$\hbar(\varphi) := \lim_{\epsilon \searrow 0} \limsup_{k \to \infty} \frac{\log^+ b_\epsilon(\varphi^k)}{k} \in [0, \infty].$$

Note: $\hbar(\varphi) = \hbar(id \times \varphi; \Delta)$ where $\Delta \subset M^- \times M$ is the diagonal.

# Barcode entropy

Some formal properties:

(i) $\hbar(\varphi) = \hbar(\varphi^{-1})$ and $\hbar(\varphi) = \hbar(\psi\varphi\psi^{-1})$.

(ii) $\hbar(\varphi^k) \leq k\,\hbar(\varphi)$.

(iii) $\hbar(\varphi \times \psi) \leq \hbar(\varphi) + \hbar(\psi)$.

(iv) $\hbar(\varphi; L)$ is lower semi-continuous in $L$ with respect to the Hofer metric.

Topological entropy:

(i) $\mathrm{h_{top}}(\varphi) = \mathrm{h_{top}}(\varphi^{-1})$ and $\mathrm{h_{top}}(\varphi) = \mathrm{h_{top}}(\psi\varphi\psi^{-1})$.

(ii) $\mathrm{h_{top}}(\varphi^k) = k\,\mathrm{h_{top}}(\varphi)$.

(iii) $\mathrm{h_{top}}(\varphi \times \psi) = \mathrm{h_{top}}(\varphi) + \mathrm{h_{top}}(\psi)$.

# Main results

**Theorem A**

*Let $L$ be a closed monotone Lagrangian submanifold with minimal Chern number $N_L \geq 2$ in a symplectic manifold $M$ and let $\varphi\colon M \to M$ be a compactly supported Hamiltonian diffeomorphism. Then*

$$\hbar(\varphi; L) \leq \mathrm{h}_{\mathrm{top}}(\varphi).$$

As a consequence, $\hbar(\varphi; L) < \infty$.

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$$\hbar(\varphi; L) \leq \mathrm{h_{top}}(\varphi).$$

As a consequence, $\hbar(\varphi; L) < \infty$. Since $\hbar(id \times \varphi; \Delta) = \hbar(\varphi)$ and $\mathrm{h_{top}}(id \times \varphi) = \mathrm{h_{top}}(\varphi)$, we have:

**Corollary A**

*Let $\varphi\colon M \to M$ be a Hamiltonian diffeomorphism of a closed monotone symplectic manifold $M$. Then*

$$\hbar(\varphi) \leq \mathrm{h_{top}}(\varphi).$$

# Main results

A compact invariant (hyperbolic) set $K$ of $\varphi$ is called *locally maximal* if there exists a neighborhood $U \supset K$ such that $K = \bigcap_{k \in \mathbb{Z}} \varphi^k(U)$.

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**Theorem B**

*Let $\varphi\colon M \to M$ be a Hamiltonian diffeomorphism of a closed monotone symplectic manifold $M$ and let $K \subset M$ be a locally maximal hyperbolic subset. Then*

$$\hbar(\varphi) \geq \mathrm{h}_{\mathrm{top}}(\varphi|_K).$$

Example: Smale's horseshoe is a locally maximal hyperbolic set.

Katok-Hasselblatt : $\mathrm{h}_{\mathrm{top}}(\varphi|_K) = \limsup_{k\to\infty} \frac{\log \left| \operatorname{Fix} \varphi^k|_K \right|}{k}$ where $K$ is locally maximal and hyperbolic.

Katok (1980): On closed surfaces

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**Theorem C**

*Let $\varphi\colon M \to M$ be a Hamiltonian diffeomorphism of a closed surface $M$. Then*

$$\hbar(\varphi) = \mathrm{h_{top}}(\varphi).$$

Proof of Theorem C: Corollary A + Theorem B + Katok's result.

Corollary A: $\hbar(\varphi) \leq \mathrm{h_{top}}(\varphi)$.

Theorem B: $\hbar(\varphi) \geq \mathrm{h_{top}}(\varphi|_K)$.

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Corollary A: $\hbar(\varphi) \leq \mathrm{h_{top}}(\varphi)$.

Theorem B: $\hbar(\varphi) \geq \mathrm{h_{top}}(\varphi|_K)$.

Remark: In Theorem B (and C), it is essential that the Floer complex is generated by all fixed points (or all intersections in the Lagrangian case).

# Previous works

Reeb flows and symplectomorphisms:

Frauenfelder-Schlenk (2006), Macarini-Schlenk (2011), Alves et al. (2014, ...), Dahinden (2018, 2021).

Hamiltonian diffeomorphisms in dimension 2:

Humilière (2017): Let $L \subset S^2$ be an equator. There exists $C > 0$ such that for all $\varphi \in Ham(S^2)$, $\limsup_{k\to\infty} d_H(L, \varphi^k(L))/k \leq C\, \mathrm{h}_{\mathrm{top}}(\varphi)$.

Khanevsky (2021): Let $L \subset \Sigma_{g\geq 1}$ be simple and non-contractible. For all $N > 0$, there exists $L_N \subset \Sigma_{g\geq 1}$ such that every $\varphi \in Ham(\Sigma_{g\geq 1})$ with $\varphi(L) = L_N$ has $\mathrm{h}_{\mathrm{top}}(\varphi) \geq N$.

# Previous works

Chor-Meiwes (2021): For all $N > 0$, there exists open dense subset $U$ of $\big(Ham(\Sigma_{g\geq 2}), d_H\big)$ such that $\mathrm{h_{top}}|_U \geq N$.

Alves-Meiwes (2021): The topological entropy $\mathrm{h_{top}} : \big(Ham(\Sigma), d_H\big) \to \mathbb{R}$ is lower semi-continuous.

Note: By Theorem C, all of these results hold for the absolute barcode entropy as well. We cannot prove directly any them at the moment.

# Barcodes

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Setup: Suppose that $L \pitchfork \varphi(L)$. The Lagrangian Floer complex

$$(\mathrm{CF}(L, \varphi(L)), d_{Fl})$$

is generated by all (capped) intersections $L \cap \varphi(L)$ over the universal Novikov field $\Lambda_{\mathbb{F}_2}$ and filtered by the action $\mathcal{A}$ (the Floer differential $d_{Fl}$ strictly decreases the action $\mathcal{A}$).

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Note that both $d_{Fl}$ and $\mathcal{A}$ depend on other data (the Hamiltonian $H$ generating $\varphi$, the complex structure $J$ and the choice of cappings). We implicitly make these choices.

# Barcodes

Usher-Zhang: A basis $\Sigma = \{\alpha_i, \gamma_j, \eta_j\}$ of $\mathrm{CF}(L, \varphi(L))$ over $\Lambda_{\mathbb{F}_2}$ is called a *singular value decomposition* if

-$d_{Fl}(\alpha_i) = 0$ and $d_{Fl}(\gamma_j) = \eta_j$,

-$\Sigma$ is orthogonal (in the non-Archimedean sense).

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-$\Sigma$ is orthogonal (in the non-Archimedean sense).

Note: A subset $\{\beta_i\} \subset \mathrm{CF}(L, \varphi(L))$ is called *orthogonal*, if

$$\mathcal{A}(\sum \lambda_i \beta_i) = \max \mathcal{A}(\lambda_i \beta_i)$$

for all $\lambda_i \in \Lambda_{\mathbb{F}_2}$. For example, suppose that $\Lambda_{\mathbb{F}_2} = \mathbb{F}_2$ and all capped intersections have distinct actions, then $\{\beta_i\}$ is orthogonal if and only if $\mathcal{A}(\beta_i)$ are distinct. Similarly, if $\mathcal{A}(\beta_1) < \mathcal{A}(\beta_2)$, then $\{\beta_2, \beta_1 + \beta_2\}$ is not orthogonal. Roughly speaking, we don't want "cancellations".

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The *barcode* $\mathcal{B}$ *of* $(\mathrm{CF}(L, \varphi(L), d_{Fl})$ is the multiset formed by the finite bars $\mathcal{A}(\gamma_j) - \mathcal{A}(\eta_j)$ together with $\dim_{\Lambda_{\mathbb{F}_2}} \mathrm{HF}(L, \varphi(L))$ many $\infty$-bars.

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<u>Usher-Zhang:</u>

(i) $(\mathrm{CF}(L, \varphi(L), d_{Fl})$ admits a singular value decomposition.

(ii) Its barcode $\mathcal{B}$ only depends on $\varphi$ and $L$.

(iii) If $d_H(\tilde{L}, L) < \delta < \epsilon/2$, then $b_\epsilon(L, \varphi(L)) \leq b_{\epsilon-2\delta}(\tilde{L}, \varphi(L))$.

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<u>Note:</u>

(i) If $L \pitchfork \varphi(L)$, then $b_\epsilon(L, \varphi(L)) \leq |L \cap \varphi(L)|$.

(ii) Since $\dim_{\Lambda_{\mathbb{F}_2}} \mathrm{HF}(L, \varphi(L))$ does not depend on $\varphi$, if $\hbar(\varphi, L) \neq 0$, then the growth comes from finite bars.

# Proof of Theorem A

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Yomdin's theorem: Let $\varphi : M \to M$ be a $C^\infty$-diffeomorphism and $N \subset M$ be a compact submanifold. Then

$$\limsup_{k\to\infty} \frac{\log \mathrm{vol}(\varphi^k(N))}{k} \leq \mathrm{h}_{\mathrm{top}}(\varphi).$$

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Proof of Theorem A: Suppose that $\hbar(\varphi; L) \neq 0$ and let $0 < \alpha < \hbar(\varphi; L)$.

Step 1: Set $L^k := \varphi^k(L)$. Let $\epsilon > 0$ and $k_i \to \infty$ such that

$$const\, 2^{k_i \alpha} \leq b_\epsilon(L, L^{k_i}).$$

By Yomdin's theorem, it suffices to show that

$$const\, 2^{k_i \alpha} \leq \mathrm{vol}(L^{k_i}).$$

# Proof of Theorem A

Step 2: (Crofton's inequality)

Let $N \subset M$ be a closed submanifold and let $B$ be a compact manifold.

Let $\Psi : B \times N \to M$ be a submersion (onto its image) with $\Psi|_{s \times N}$ an embedding for all $s \in B$. Set $N_s := \Psi(s, N)$.

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**Lemma**

*Let $\tilde{N} \subset M$ be a closed submanifold with $\operatorname{codim} \tilde{N} = \dim N$. We have*

$$\int_B |N_s \cap \tilde{N}| \, ds \leq const \, \operatorname{vol}(\tilde{N}).$$

*where the constant does not depend on $\tilde{N}$.*

# Proof of Theorem A:

Step 3: (Lagrangian tomograph)

Let $\Psi : B^d \times L \to T^*L \subset M$ be a submersion (onto its image) with

(i) $L_s := \Psi(s, L)$ is an embedded Lagrangian for all $s \in B^d$,

(ii) $d_H(L_s, L) < \delta < \epsilon/2$ for all $s \in B^d$.

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By Hofer stability of $b_\epsilon(L, L^{k_i})$, we have

$$const\, 2^{k_i\alpha} \leq b_\epsilon(L, L^{k_i}) \leq b_{\epsilon-2\delta}(L_s, L^{k_i}) \leq |L_s \cap L^{k_i}|$$

for almost all $s \in B^d$.

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for almost all $s \in B^d$. Then, by Crofton's inequality, we have

$$const\, 2^{k_i\alpha} \leq \mathrm{vol}(L^{k_i}) \implies \alpha \leq \mathrm{h_{top}}(\varphi).$$

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**Lemma**

*Suppose that $\varphi$ has $p$ $\epsilon$-isolated fixed points. Then $b_\epsilon(\varphi) \geq p/2$.*

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**Lemma**

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Katok-Hasselblatt: $\mathrm{h_{top}}(\varphi|_K) = \limsup_{k\to\infty} \frac{\log \left| \mathrm{Fix}\, \varphi^k|_K \right|}{k}$.

It suffices to show that there exists $\epsilon_K > 0$ such that all periodic points contained in $K$ are $\epsilon_K$-isolated.

# Proof of Theorem B

Step 2:

Ginzburg-Gürel (2018): (Crossing Energy Theorem)

Let $\tilde{K} \subset \tilde{U} \subset S^1 \times M$ be an isolating neighborhood. There exist $\epsilon > 0$ such that all Floer cylinders $u$ that are

(i) asymptotic to a $k$-periodic point contained in $K$,

(ii) $\tilde{u} \not\subset \tilde{U}$

have energy $E(u) > \epsilon$.

Remark: Here $\epsilon > 0$ is independent of $k$.

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Remark: Here $\epsilon > 0$ is independent of $k$.

Next we focus on Floer cylinders that are contained in $\tilde{U}$.

# Proof of Theorem B

Step 3:

A set $\{z_i \mid i \in \mathbb{Z}_k\} \subset M$ is called an *$\eta$-pseudo-orbit of $\varphi$*, if

$$d(\varphi(z_i), z_{i+1}) < \eta \quad \text{for all} \quad i \in \mathbb{Z}_k.$$

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Anosov Closing Lemma $\implies$ There exists $U \supset K$, $\eta > 0$ and $\delta > 0$ such that all $\eta$-pseudo-orbits contained in $U$ are uniquely $\delta$-shadowed by a true orbit in $K$.

Namely, if $\{z_i \,|\, i \in \mathbb{Z}_k\} \subset U$ is an $\eta$-pseudo-orbit, then there exists a unique periodic point $x \in K$ of $\varphi$ such

$$d(\varphi^i(x), z_i) < \delta \quad \text{for all} \quad i \in \mathbb{Z}_k.$$

# Proof of Theorem B

Let $u : \mathbb{R} \times S^1_k \to M$ be a Floer cylinder contained in $\tilde{U}$. We have

$$\{u(\pm\infty, i) \mid i \in \mathbb{Z}_k\} \subset K.$$

(i) $E(u)$ is sufficiently small $\implies$ $||\partial_s u||$ is small point wise.

(ii) $||\partial_s u||$ is small $\implies$ there exists $\eta > 0$, independent of $k$, such that $\hat{z}(s) := \{u(s, i) \mid i \in \mathbb{Z}_k\} \subset U$ is an $\eta$-pseudo-orbit for all $s \in \mathbb{R}$.

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(iii) Anosov Closing Lemma $\implies$ if $\eta > 0$ is sufficiently small, then $\hat{z}(s)$ is shadowed by a unique periodic point $w(s) \in K$.

(iv) $w(s)$ depends continuously on $s$ $\implies$ $u(-\infty, t) = u(+\infty, t)$.

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(ii) $||\partial_s u||$ is small $\implies$ there exists $\eta > 0$, independent of $k$, such that $\hat{z}(s) := \{u(s, i) \,|\, i \in \mathbb{Z}_k\} \subset U$ is an $\eta$-pseudo-orbit for all $s \in \mathbb{R}$.

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(iv) $w(s)$ depends continuously on $s \implies u(-\infty, t) = u(+\infty, t)$.

Conclusion: There exists $\epsilon_K > 0$ such that if $\tilde{u} \subset \tilde{U}$ and $E(u) \leq \epsilon_K$, then $u$ is constant.

The End