Invariant submanifolds for conformal symplectic dynamics

Marie-Claude Arnaud & Jacques Fejoz

March 4th 2022

Setting

Let $(\mathcal{M}^{2d}, \omega)$ be a symplectic manifold. By a *conformal symplectic dynamics*, we mean

- a diffeomorphism $f : \mathcal{M} \circlearrowleft$ such that $f^*\omega = a\,\omega$ with $a \in (0,1) \cup (1, +\infty)$.
- or a complete vector field X such that $L_X\omega = \alpha\,\omega$, with $\alpha \neq 0$.

$(T^*Q, -d\lambda)$ Z_λ Liouville

ok

flow very simple

zero section Lagrangian

Some remarks

$\mathcal{A}$ is not compact. Indeed

$$\mathrm{Vol}(\mathcal{A}) = \mathrm{Vol}(f(\mathcal{A})) = a\,\mathrm{Vol}(\mathcal{A})$$

If X conformal.

$$L_X\omega = \alpha\omega$$

$$\| \qquad\qquad \Downarrow$$

$$d(i_X\omega) \qquad \omega \text{ is exact. } = \frac{1}{\alpha}\, d(i_X\omega)$$

Birkhoff studied dissipative twist maps of $\mathbb{T}\times\mathbb{R}$ that can have very complicated attractors (indecomposable [illegible])

Isotropy: some preliminary remarks

- A closed surface that is invariant by a conformal symplectic dynamics has to be isotropic;
- R.C. Calleja, A. Celletti & R. de la Llave proved in 2013 that if a C^1 conformal dynamics has a C^1 invariant torus on which the dynamics is C^1 conjugate to a rigid rotation, then this torus is isotropic.

Isotropy: an example

Proposition

There exists a conformal symplectic vector field X on a 4-dimensional symplectic manifold $(\mathcal{M}, \omega)$,with a 3-dimensional invariant submanifold $\mathcal{L}$. Moreover, the submanifold $\mathcal{L}$ is the global attractor for the flow (φ_t) of X, $(\varphi_{t|\mathcal{L}})$ is conjugate to the suspension of an Anosov automorphism of $\mathbb{T}^2$ with 2-dimensional stable and unstable foliations, and $(\varphi_{t|\mathcal{L}})$ is transitive with entropy equal to $|\alpha|$, where α is the conformality rate of X.

Isotropy and entropy

Theorem

Let $f : \mathcal{M} \circlearrowleft$ be a C^3 conformal symplectic diffeomorphism such that $f^*\omega = a\omega$ with $a > 1$. Suppose that $\mathcal{N}$ is an invariant C^3 submanifold such that the induced form $\omega_{|\mathcal{N}}$ on $\mathcal{N}$ has constant rank 2ℓ. Then

$$\mathrm{ent}\, f_{|\mathcal{N}} \geqslant \ell \ln a;$$

hence, when $\mathrm{ent}\, f_{|\mathcal{N}} < \ln a$, $\mathcal{N}$ is isotropic.

This happens when $f_{|\mathcal{N}}$ is minimal i.e. every orbit is dense, e.g. when $f_{|\mathcal{N}}$ is C^0 conjugated to a minimal rotation

This theorem can be used when f is C^0 conjugated to a minimal rigid rotation: entropy $= 0$

Rq Yomdin theory:

$$\mathrm{ent}\, f_{|\mathcal{N}} + \text{“local entropy”} \geqslant \ell \ln a.$$

$\to 0$ when the dynamics in C^∞.

Liouville class

We asssume that $(\mathcal{M}, \omega) = (T^*\mathcal{Q}, -d\lambda)$ and let $\pi : T^*\mathcal{Q} \to \mathcal{Q}$ be the canonical projection.

Definition

Let $\mathcal{L}$ be a Lagrangian submanifold of $T^*\mathcal{Q}$ that is homotopic to the zero section $\mathcal{Z}$. Then the restriction of π to $\mathcal{L}$ induces an isomorphism between $H^1(\mathcal{L}, \mathbb{R})$ and $H^1(\mathcal{Q}, \mathbb{R})$. Denoting by $j_\mathcal{L} : \mathcal{L} \hookrightarrow T^*\mathcal{Q}$ the canonical injection, the Liouville class of the submanifold $\mathcal{L}$ is the cohomological class

$$[\mathcal{L}] = \left[\left(\pi_{|\mathcal{L}}\right)_* (j_\mathcal{L}^* \lambda) \right] \in H^1(\mathcal{Q}, \mathbb{R}).$$

$[\mathcal{L}] = 0 \iff \mathcal{L}$ is exact lagrangian

Liouville class and conformal dynamics (1)

Proposition

Let $f : T^*\mathcal{Q} \circlearrowleft$ *be a conformal diffeomorphism that is homotopic to* $\mathrm{Id}_{T^*\mathcal{Q}}$*. Then* $\eta = f^*\lambda - a\lambda$ *is a closed 1-form.*
Let $\mathcal{L} \subset T^*\mathcal{Q}$ *be a Lagrangian submanifold that is homotopic to the zero section. Then*

$$[f(\mathcal{L})] = a[\mathcal{L}] + \pi_*[\eta].$$

$a \neq 1$

Corollary

Let $f : T^*\mathcal{Q} \circlearrowleft$ *be a CS-diffeomorphism that is homotopic to* $\mathrm{Id}_{T^*\mathcal{Q}}$*. Then there is only one Liouville class that we denote by* $[\ell_f]$*, that a homotopic to the zero section and* f*-invariant submanifold may have.*

Lagrangian.

Liouville class and conformal dynamics (2)

Theorem

conformal exact symplectic $f^*\lambda - a\lambda = dS$.

If $f: T^*\mathcal{Q} \circlearrowleft$ is a λ CES diffeomorphism that is CS-isotopic to $\mathrm{Id}_{T^*\mathcal{Q}}$ and $\mathcal{L}$ is a Lagrangian submanifold that is isotopic to the zero section among the Lagrangian submanifolds of $T^*\mathcal{Q}$ such that $\bigcup_{k\in\mathbb{Z}} f^k(\mathcal{L})$ is relatively compact, then $\mathcal{L}$ is exact.

If not, $[\mathcal{L}] \neq 0$. $[f^k\mathcal{L}] = a^k[\mathcal{L}] \xrightarrow{k\to\pm\infty} \infty$.

Question if $[\mathcal{L}_n] \xrightarrow{n\to\infty} \infty$, can I say that ~~there~~ $\bigcup \mathcal{L}_n$ is not relatively compact?

$\to$ YES. simple case: $\exists \eta$ with no zero st $[\eta] = [\mathcal{L}]$.

$[a^k\eta] = [f^k\mathcal{L}]$ $\quad f^k(\mathcal{L}) \cap \mathrm{graph}(a^k\eta) \neq \emptyset$.

Uniqueness (1)

We assume that $f : T^*\mathcal{Q} \circlearrowleft$ is CS and CH (conformally hamltonianly isotopic) to $\mathrm{Id}_{T^*\mathcal{Q}}$, i.e. the isotopy is given by $i_{X_t}\omega = \alpha_t\lambda + dH_t$. We recall that Viterbo introduced a spectral distance (γ-distance) on the set of Lagrangian submanifolds that are H-isotopic to the zero-section.

Proposition

Let $\mathcal{L}$, $\mathcal{L}'$ be two H-isotopic to the zero section submanifolds of $T^*\mathcal{Q}$. Let (ϕ_t) be an isotopy of exact conformal symplectic diffeomorphisms of $T^*\mathcal{Q}$ such that $\phi_0 = \mathrm{Id}_{T^*\mathcal{Q}}$ and $\phi_t^*\omega = a(t)\omega$. Then

$$\gamma(\phi_t(\mathcal{L}), \phi_t(\mathcal{L}')) = a(t)\gamma(\mathcal{L}, \mathcal{L}').$$

Corollary

Let $f : \mathcal{M} \circlearrowleft$ be a CS diffeomorphism that is CH-isotopic to $\mathrm{Id}_{T^*\mathcal{Q}}$. Then there exists at most one H-isotopic to the zero section submanifold of $T^*\mathcal{Q}$ that is invariant by f.

Uniqueness (2)

Theorem

Let $f : T^*\mathbb{T}^n \circlearrowleft$ be a CES diffeomorphism that is CH-isotopic to $\mathrm{Id}_{T^*\mathbb{T}^n}$. Then there exists at most one H-isotopic to the zero section submanifold $\mathcal{L}$ such that

$$\bigcup_{k\in\mathbb{Z}} f^k(\mathcal{L}) \quad \textit{is} \quad \textit{relatively} \quad \textit{compact.}$$

Hence when it exists, $\mathcal{L}$ is invariant by the f.

if $f^k(\mathcal{L}) \neq \mathcal{L}$ for one k, then $\gamma(f^{\ell+k}(\mathcal{L}), f^{\ell}(\mathcal{L})) \underset{\ell\to\pm\infty}{\longrightarrow} +\infty$

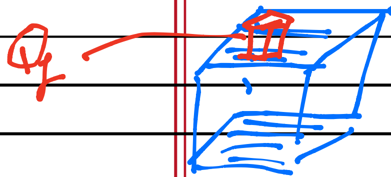
Shelukhin: if $\mathcal{K} \subset \{(q,p),\ \|p\| \leq R\}$, $\exists C$, $\forall \mathcal{L}, \mathcal{L}'$ Ham isot to Z, $\gamma(\mathcal{L}, \mathcal{L}') \leq C$.

$\Rightarrow$ if $\bigcup_{k\in\mathbb{Z}} f^k(\mathcal{L})$ is relatively compact, $\forall k$, $f^k(\mathcal{L}) = \mathcal{L}$.

PROOF OF:

Thm $f : M \circlearrowleft C^3$, ω closed 2form, rank $= 2\ell$. $f^*\omega = a\omega$. $a > 1$

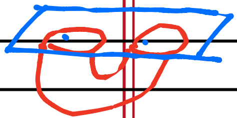
? $\mathrm{Ent} f > \ell \ln a$?



$\mathcal{F}^0$ characteristic

adapted charts $\mathcal{W}_i$ $\qquad Q_j \subset \mathcal{W}_i$

leaf of $\mathcal{F}^0_{|Q_j} \hookrightarrow$ leaf of $\mathcal{F}\,\mathcal{W}_i$ injective.



$f(Q_j) \subset \mathcal{W}_k$ $\qquad$ leaf of $\mathcal{F}^0_{|Q_j} \xhookrightarrow{f}$ leaf of $\mathcal{W}_k$ injective

$\exists$ piece of submanifold $\subset Q_j$ that meets each leaf of $\mathcal{F}^0_{|Q_j}$ at most once. for all k, $f(S) \cap Q_k$ meets also each leaf of $\mathcal{F}^0_{|Q_k}$ at most once.

$$|\Omega^\ell(f^k(S))| = a^{k\ell}|\Omega(S)| \leq \underbrace{\#\{(j_1, \dots, j_k);\ f^k(S) \cap f^{k-1}(Q_{j_1}) \cap \dots \cap Q_{j_k} \neq \emptyset\}}_{N_k} \max\{\Omega^\ell(Q_j),\ 1 \leq j \leq N\}.$$

$\hookrightarrow$ symplectic reduction of Q_j!

$$\mathrm{Ent} f \geq \lim \frac{1}{k} \ln(N_k).$$
$$\geq \ell \ln a + \frac{\ln \Omega(S)}{k} - \frac{T}{k}$$

vector field

e^{ss} contraction

$\frac{1}{\lambda^2}$

e^u, e^{ss} conjugate vectors' for ω: $\omega(e^u, e^{ss}) = 1$.

$\omega(\lambda e^u, \frac{1}{\lambda^2} e^{ss}) = \frac{1}{\lambda}$.

(e^s, X) $\omega(\frac{1}{\lambda} e^s, X) = \frac{1}{\lambda}$